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Nonlinear electrostatic waves in collisionless plasmas

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We report results concerning small amplitude Bernstein-Greene-Kruskal (BGK) waves, which are exact undamped traveling wave solutions of the nonlinear Vlasov-Poisson-Ampère equations for collisionless plasmas. Building upon previous work, we first develop a simple but powerful formalism that facilitates a methodical investigation of the types and properties of small amplitude BGK plasma waves that can exist near a given collisionless plasma equilibrium. Using this formalism, we then show that any physically relevant spatially uniform plasma equilibrium supports nonlinear *spatially periodic* BGK waves that are described by the Vlasov dispersion relation in the small amplitude limit. We demonstrate also that these equilibria are characterized by a discrete set of critical velocities $v_c^{(i)}$, $i=1,2,\ldots$, at which BGK *solitary waves* of vanishingly small amplitude can propagate in the plasma. The existence of these exact nonlinear spatially periodic and solitary wave solutions illustrates the fundamental incompleteness of the linear Vlasov-Landau theory of plasma waves since, by virtue of particle trapping, these nonlinear waves neither damp nor grow even when their amplitude is arbitrarily small.

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I. INTRODUCTION

The nonlinear Vlasov-Maxwell model of kinetic theory [1] presently provides the most appropriate theoretical description for plasmas in the collisionless regime, wherein particle-particle interactions have a negligible effect upon the collective plasma dynamics. Due to the analytical difficulties associated with nonlinearity, however, the theory of processes in collisionless plasmas has in practice been based predominantly upon Landau's classic 1946 analysis [2] of the linearized approximation of this nonlinear model. Insights gained from analysis of the linearized model have contributed greatly to the understanding of plasma dynamics in a wide variety of settings ranging from laboratory plasmas in thermonuclear fusion devices [3] and particle accelerating machines, for instance, to naturally occurring near-earth, interplanetary, solar, and astrophysical plasmas [4]. When a small amplitude electrostatic disturbance is introduced into a spatially uniform host plasma, Landau's analysis predicts, for plasmas with more or less thermal distributions of particle velocities, that the electric field associated with the disturbance should decay to zero asymptotically with time. This phenomenon, known as collisionless wave damping or "Landau damping," has been substantiated by experiments [5,6] in laboratory plasmas; moreover, Dawson [7] has given a simple physical explanation of the phenomenon in terms of the resonant transfer of energy between the wave and the particles of the plasma.

On the other hand, neither Landau's linearized analysis nor any of the other prominent and physically equivalent linear analyses [8,9] tell the complete story of small amplitude plasma waves. As early as 1949, for instance, Bohm and Gross [10] recognized the possibility of nonlinear traveling waves of arbitrarily small but *constant* amplitude. Undamped waves of this type are distinguished by the existence of plasma particles that are trapped within the electrostatic potential wells formed by the wave. These trapped particles modify the space-averaged plasma distribution functions in such a way as to inhibit wave damping. Particle trapping is ignored by traditional linear theories, which assume that the presence of small amplitude waves has a negligible effect upon the space-averaged distribution functions. In 1957, Bernstein, Greene, and Kruskal [11] formalized the methods of Bohm and Gross and characterized a class of basic exact nonlinear solutions of the Vlasov-Maxwell equations, which have become known variously as BGK modes, BGK waves, or BGK equilibria. These authors did not focus on small amplitude waves however, and perhaps this explains why many physicists believe that the linear theory offers a complete description of small amplitude plasma waves, even though the long-standing work of Bohm and Gross conclusively demonstrates otherwise.

In this paper we report the results of a systematic study of these oft neglected nonlinear plasma waves of arbitrarily small yet constant amplitude. We shall apply the label "BGK" to these waves since this has become practiced usage for large amplitude waves of the same type. But, while the synthesis of Bernstein, Greene, and Kruskal was an analytical milestone, the importance of the 1949 paper of Bohm and Gross cannot be overstated, for it provided the seeds for the entire subsequent development of the field of undamped waves and, in fact, already contained many of the important physical conclusions. Recent work by Holloway [12] and by Holloway and Dorning [13] has provided certain missing de-

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tails concerning the mathematical description of these waves in the small amplitude limit and in particular, gives a rigorous nonlinear characterization of the particle distribution functions. In this paper we shall recast some of the developments of that recent work into a simple and more powerful formalism, which we then shall use to methodically investigate the types and properties of small amplitude BGK plasma waves, both spatially periodic and aperiodic, that can exist nearby physically relevant collisionless plasma equilibria. We have previously summarized these results very briefly elsewhere [14].

After introducing the basic one-dimensional model for electrostatic processes in Sec. II, we briefly review the linear theory of electrostatic traveling waves in Sec. III. In Sec. IV we introduce the essential ideas of Bohm and Gross and of Bernstein, Greene, and Kruskal and discuss the representation scheme for the plasma distribution functions developed from these ideas by Holloway and Dorning. This representation scheme is crucially important since it reduces the problem of searching for traveling wave solutions of a set of three partial differential equations to a much simpler search for solutions of a single ordinary differential equation. In Sec. V we develop a simple method for solving this equation, which obviates much of the complicated nonlinear analysis carried out in Refs. [12] and [13], and we then apply this approach to determine the most important general classes of near-equilibrium collisionless plasma traveling wave states and establish their properties. We illustrate these general results in the case of the practically important thermal e^{-}-p^{+} plasma in Sec. VI and in Sec. VII present a general argument that shows that small amplitude undamped waves, both spatially periodic and solitary, can exist near all physically relevant plasma equilibria. Finally, in Sec. VIII we comment briefly on the possible relevance of these solutions to waves in physical plasmas, particularly plasmas that evolve toward time-asymptotic states following nonlinear Landau damping.

II. ELECTROSTATIC PROCESSES

A plasma can support an enormous variety of processes such as transverse electromagnetic waves, in which case the plasma may be adequately treated as a dielectric medium, as well as longitudinal or compressional waves, which often exhibit fundamentally nonlinear features. We shall be concerned here only with the latter. Specifically, the one-dimensional model we shall study describes a commonly occurring physical situation in which a rarefied plasma is embedded in an external magnetic field, in which case the longitudinal (along $\vec{B}$) degrees of freedom may be studied apart from their very complicated interactions with other degrees of freedom that are involved in general three-dimensional plasma motion. For a plasma in a field $\vec{B}=B_0\hat{\vec{x}}$; the Vlasov-Maxwell model can be reduced to the one-dimensional Vlasov-Poisson-Ampère system of equations

$$\frac{\partial f_\alpha}{\partial t}+u\frac{\partial f_\alpha}{\partial x}+\frac{q_\alpha}{m_\alpha}E\frac{\partial f_\alpha}{\partial u}=0 , \tag{1}$$

$$\frac{\partial E}{\partial x}=4\pi\sum_\alpha q_\alpha\int du\, f_\alpha , \tag{2}$$

$$-\frac{\partial E}{\partial t}=4\pi\sum_\alpha q_\alpha\int du\, u\, f_\alpha , \tag{3}$$

where $f_\alpha=f_\alpha(x,u,t)$ is the distribution function for particle species α, $\alpha=1,2,\ldots,N$, and $E=E(x,t)$ is the self-consistent longitudinal electric field.

The simplest solutions of Eqs. (1)–(3) describe spatially homogeneous equilibrium states of the plasma, usually called "Vlasov equilibria." The distribution functions $f_\alpha(x,u,t)=F_\alpha(u)$ of a spatially uniform Vlasov equilibrium (we shall always denote equilibria by capital letters) must give zero charge and current densities

$$\rho=\sum_\alpha q_\alpha\int du\, F_\alpha=0 , \tag{4}$$

$$j=\sum_\alpha q_\alpha\int du\, u\, F_\alpha=0 , \tag{5}$$

which imply, through Eqs. (2) and (3), a vanishing electric field. In an actual physical system the states corresponding to these equilibria in fact represent "metaequilibria" that should evolve, given enough time, toward thermal equilibrium. But in many naturally occurring, highly rarefied plasmas, the time scale over which thermal equilibrium is attained is very long compared to that over which collective plasma processes occur. Thus, on the time scales relevant to these collective phenomena, which include stationary plasma oscillations as well as traveling waves, it is appropriate to consider the Vlasov equilibria as true stationary states of the plasma.

Throughout this paper we shall be concerned with plasma states that are close to Vlasov equilibria. In particular, we shall consider near-equilibrium states of traveling wave form, i.e., $E(x,t)=E(x-vt)$, $f_\alpha(x,u,t)=f_\alpha(x-vt,u-v)$, where v is the wave phase velocity. Plasmas far from equilibrium are of course also important, but near equilibrium states are of special interest since they can arise as the result of weak perturbations of an equilibrium plasma and are therefore relevant to wavelike processes observed in many natural plasma environments.

III. LINEAR THEORY

The conclusions of the analysis of the linearized form of Eqs. (1)–(3) are well known. We review them here for completeness and also because we shall find later that there are very important differences between the conclusions of this linear analysis and recent rigorous nonlinear analyses. Linearization about a particular Vlasov equilibrium $F_\alpha(u)$ is performed by writing

$$f_\alpha(x,u,t)=F_\alpha(u)+h_\alpha(x,u,t) , \tag{6}$$

where the h_α represent the presumed small deviations of the distribution functions from their equilibrium values F_α. One of the resulting terms in the Vlasov equation, $(q_\alpha/m_\alpha)E(\partial h_\alpha/\partial u)$, is then dropped since it is nonlinear as E itself depends linearly on h_α through Poisson's equation. The set of linearized equations so obtained are

$$\frac{\partial h_\alpha}{\partial t}+u\frac{\partial h_\alpha}{\partial x}+\frac{q_\alpha}{m_\alpha}E\frac{dF_\alpha}{du}=0 , \tag{7}$$

$$\frac{\partial E}{\partial x}=4\pi\sum_\alpha q_\alpha\int du\, h_\alpha , \tag{8}$$

and

$$-\frac{\partial E}{\partial t}=4\pi\sum_\alpha q_\alpha\int du\, u\, h_\alpha . \tag{9}$$

Clearly, for the neglected term $(q_\alpha/m_\alpha)E(\partial h_\alpha/\partial u)$ to be small compared to the term $(q_\alpha/m_\alpha)E(dF_\alpha/du)$, which is retained in the Vlasov equation, it is necessary that

$$\left|\frac{\partial h_\alpha}{\partial u}\right| \ll \left|\frac{dF_\alpha}{du}\right| \tag{10}$$

for all x, u, and t.

In his 1946 analysis [2] Landau solved the initial value problem for Eqs. (7)–(9). His essential result was an expression for the time-asymptotic value of the electric potential

$$\varphi_k(t) \underset{t\to\infty}{\sim} c_k(\bar{\lambda}_k)e^{\bar{\lambda}_k t} , \tag{11}$$

where $c_k(\bar{\lambda}_k)$, a constant, depends upon the initial perturbation of the plasma and $\bar{\lambda}_k$ is that root of the Landau dispersion relation

$$D_F(k,\lambda)=1-\frac{4\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\int_L du\frac{F'_\alpha(u)}{u-i\lambda/k}=0 \tag{12}$$

that has least negative real part. The Landau contour L is depicted in Fig. 1. Whether $\mathrm{Re}(\bar{\lambda}_k)$ is positive or negative depends solely upon the distribution functions $F_\alpha(u)$ of the Vlasov equilibrium, since these determine the function $D_F(k,\lambda)$. The linear analysis predicts exponential growth of the electric field if $\mathrm{Re}(\bar{\lambda}_k)>0$ and exponential damping if $\mathrm{Re}(\bar{\lambda}_k)<0$. The latter, which results for all monotonically decreasing equilibria $[\sum_\alpha(q_\alpha^2/m_\alpha)F_\alpha(u)$ decreasing with increasing $|u|$, as in a thermal equilibrium plasma], is the well-known phenomenon of "Landau damping."

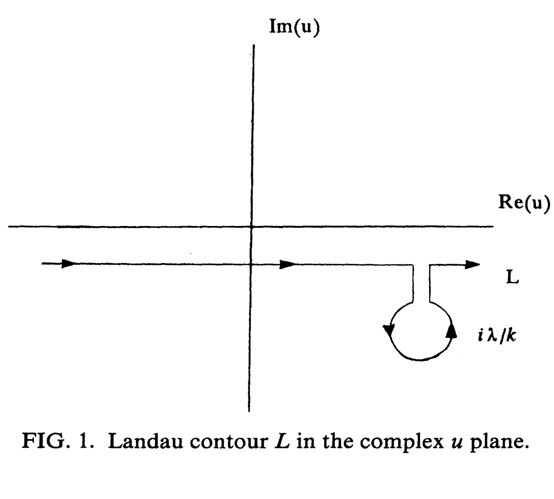

FIG. 1. Landau contour L in the complex u plane.

Prior to Landau's work, Vlasov [1] had obtained on the basis of the linearized equations a different dispersion relation

$$1-\frac{4\pi}{k^2}\sum_\alpha\frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int du\frac{F'_\alpha(u)}{u-\omega/k}=0 , \tag{13}$$

which describes *undamped* plasma waves and differs from Eq. (12) in that it involves the principal value P rather than the Landau contour. However, since Vlasov offered no plausible defense for his *ad hoc* introduction of this principal value, the "Vlasov dispersion relation" of Eq. (13) has long been considered irrelevant [except insofar as it arises in seeking roots of Eq. (12) that correspond to weakly damped solutions]. Nevertheless, as we shall see in Sec. V, Eq. (13) does correspond to a very important class of plasma waves.

The condition Eq. (10), necessary for the validity of the linear approximation, does not necessarily agree with the common sense notion that the linearization should be valid when the h_α are small, that is, when there is only a minor rearrangement of particles. This apparently minor point is in fact very important, for it leaves open the possibility that there exist exact nonlinear near-equilibrium states that violate the condition of Eq. (10) even as $h_\alpha\to 0$. In fact, the principle theme of this paper is that the linear theory can fail even for plasma wave states that are arbitrarily close to equilibrium. We shall develop such solutions and investigate their properties in detail in the following sections.

IV. THE BERNSTEIN-GREENE-KRUSKAL REPRESENTATION

That the linear theory does not offer a complete description *even of small amplitude plasma waves* has in fact been known since the work of Bohm and Gross [10], who discussed small amplitude waves that maintain constant amplitude as they travel. The existence of these waves depends upon a nonlinear phenomenon known as particle trapping, which is not captured by the linear theory. Bernstein, Greene, and Kruskal [11] obtained exact nonlinear solutions to the Vlasov-Poisson system by generalizing and formalizing the approach first suggested by Bohm and Gross. The essential idea is that for a uniformly translating wave $\varphi=\varphi(x-vt)$, the conserved single-particle energy $\mathcal{E}_\alpha=\frac{1}{2}m_\alpha(u-v)^2+q_\alpha\varphi(x-vt)$ can be exploited to integrate exactly the motion of the plasma particles. The self-consistent problem is then one of distributing particles over the calculated trajectories, including those corresponding to trapped particles, in such a way as to generate the assumed field $\varphi=\varphi(x-vt)$. Bohm and Gross solved the problem approximately for small amplitude waves without making specific use of the Vlasov equation. By using this equation, Bernstein, Greene, and Kruskal were able to solve the self-consistent problem without recourse to linearization. Below, in making these ideas more specific, we shall depart from the original scheme of Ref. [11] and instead develop another approach, more convenient for studying near-equilibrium plasma states, which we shall exploit in Sec. V.

We consider a plasma of N species and search for traveling wave solutions of velocity v nearby a Vlasov equilibrium characterized by distribution functions F_α. It is simplest to conduct the analysis in the wave frame where the independent variables are $\xi = x - vt$ and $w = u - v$ and the distribution functions f_α are stationary, i.e., $f_\alpha = f_\alpha(\xi, w)$. To study traveling wave states that are near equilibrium, it is convenient to resolve $f_\alpha(\xi, w)$ into even and odd parts with respect to velocity w,

$$\begin{aligned} f_\alpha^e(\xi,w) &= \tfrac{1}{2}[f_\alpha(\xi,w) + f_\alpha(\xi,-w)] , \\ f_\alpha^o(\xi,w) &= \tfrac{1}{2}[f_\alpha(\xi,w) - f_\alpha(\xi,-w)] . \end{aligned} \tag{14}$$

We then find that the full nonlinear equations (1)–(3) separate, in the wave frame, into two pairs of time-independent equations

$$w\frac{\partial f_\alpha^e}{\partial \xi} - \frac{q_\alpha}{m_\alpha}\frac{d\varphi}{d\xi}\frac{\partial f_\alpha^e}{\partial w} = 0 , \tag{15}$$

$$-\frac{d^2\varphi}{d\xi^2} = 4\pi \sum_\alpha q_\alpha \int du\, f_\alpha^e \tag{16}$$

and

$$w\frac{\partial f_\alpha^o}{\partial \xi} - \frac{q_\alpha}{m_\alpha}\frac{d\varphi}{d\xi}\frac{\partial f_\alpha^o}{\partial w} = 0 , \tag{17}$$

$$0 = 4\pi \sum_\alpha q_\alpha \int du\, u\, f_\alpha^o , \tag{18}$$

where f_α^e and f_α^o satisfy identical Vlasov equations.

The basic insight of Ref. [11] is that the conservation of the single-particle energy $\mathcal{E}_\alpha$ in the field $\varphi(\xi)$ is reflected, in the context of the Vlasov equation, by the fact that $f_\alpha^e = g_\alpha^e(\mathcal{E}_\alpha)$ and $f_\alpha^o = g_\alpha^o(\mathcal{E}_\alpha)$ are exact solutions to Eqs. (15) and (17), where g_α^e and g_α^o are any differentiable functions. In words, any distribution of particles according to their total energy $\mathcal{E}_\alpha$ automatically satisfies the Vlasov equation. Indeed, since the Vlasov operator gives the total time derivative along particle trajectories, any function of $\mathcal{E}_\alpha$ is constant along such trajectories by virtue of the invariance of $\mathcal{E}_\alpha$. We shall use this insight in seeking steady solutions of Eqs. (15)–(18) that are near the velocity-shifted equilibrium $F_\alpha^v(w) \equiv F_\alpha(w+v)$; such solutions are small amplitude traveling waves when viewed in the original frame.

Once g_α^e and g_α^o are chosen as specific functions, which we denote as $g_\alpha^{v,e}$ and $g_\alpha^{v,o}$, then the BGK representation

$$f_\alpha^e(\xi,w) = g_\alpha^{v,e}(\mathcal{E}_\alpha) \tag{19}$$

and

$$f_\alpha^o(\xi,w) = \begin{cases} g_\alpha^{v,o}(\mathcal{E}_\alpha), & w \ge 0 \\ -g_\alpha^{v,o}(\mathcal{E}_\alpha), & w \le 0 \end{cases} \tag{20}$$

immediately solves Eqs. (15) and (17), while Eqs. (16) and (18) become

$$-\frac{d^2\varphi}{d\xi^2} = 4\pi \sum_\alpha q_\alpha \int_{-\infty}^{\infty} dw\, g_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2 + q_\alpha\varphi(\xi)) , \tag{21}$$

$$0 = 8\pi \sum_\alpha q_\alpha \int_0^{\infty} dw\, w\, g_\alpha^{v,o}(\tfrac{1}{2}m_\alpha w^2 + q_\alpha\varphi(\xi)) . \tag{22}$$

Thus the search for space-dependent equilibria of Eqs. (15)–(18) near the Vlasov equilibrium $F_\alpha^v(w)$ is essentially reduced to the much simpler search for small amplitude solutions $\varphi(\xi)$ of Eqs. (21) and (22).

The judicious choice of the BGK functions $g_\alpha^{v,e}$ and $g_\alpha^{v,o}$ is crucial in this reduction. A near-equilibrium plasma state necessarily has a small electric field and satisfies $f_\alpha^e(\xi,w) \simeq F_\alpha^{v,e}(w)$ and $f_\alpha^o(\xi,w) \simeq F_\alpha^{v,o}(w)$, where $F_\alpha^{v,e}(w)$ and $F_\alpha^{v,o}(w)$ are the even and odd parts of the velocity-shifted equilibrium $F_\alpha^v(w) \equiv F_\alpha(w+v)$,

$$\begin{aligned} F_\alpha^{v,e}(w) &= \tfrac{1}{2}[F_\alpha(v+w) + F_\alpha(v-w)] , \\ F_\alpha^{v,o}(w) &= \tfrac{1}{2}[F_\alpha(v+w) - F_\alpha(v-w)] . \end{aligned} \tag{23}$$

Thus the functions $g_\alpha^{v,e}$ and $g_\alpha^{v,o}$ must be chosen carefully so that when $\varphi(\xi)=0$ the particle distribution reduces to that of the equilibrium $F_\alpha^v(w)$, for this guarantees that for small $\varphi(\xi)$ the distribution functions are close to those of the equilibrium. In the remainder of this section we shall outline complete and very flexible definitions for the functions $g_\alpha^{v,e}$ and $g_\alpha^{v,o}$, which were developed in Refs. [12] and [13]. This will set the stage for Sec. V, where we shall analyze Eqs. (21) and (22) to determine the physically important classes of small amplitude undamped nonlinear waves that can exist nearby typical plasma equilibria.

In order to define the functions $g_\alpha^{v,e}$ and $g_\alpha^{v,o}$ appropriately it is useful to first define BGK functions $G_\alpha^{v,e}(\mathcal{E}_\alpha)$ and $G_\alpha^{v,o}(\mathcal{E}_\alpha)$ that can be used, when $\varphi(\xi)=0$, to represent the equilibrium functions $F_\alpha^{v,e}(w)$ and $F_\alpha^{v,o}(w)$. The necessary definitions for $G_\alpha^{v,e}(\mathcal{E}_\alpha)$ and $G_\alpha^{v,o}(\mathcal{E}_\alpha)$ are

$$G_\alpha^{v,e}(\mathcal{E}_\alpha) = F_\alpha^{v,e}((2\mathcal{E}_\alpha/m_\alpha)^{1/2}) \tag{24}$$

and

$$G_\alpha^{v,o}(\mathcal{E}_\alpha) = F_\alpha^{v,o}((2\mathcal{E}_\alpha/m_\alpha)^{1/2}) . \tag{25}$$

When $\varphi(\xi)=0$, and therefore $\mathcal{E}_\alpha = m_\alpha w^2/2$, $F_\alpha^{v,e}(w)$ and $F_\alpha^{v,o}(w)$ are expressed using $G_\alpha^{v,e}(\mathcal{E}_\alpha)$ and $G_\alpha^{v,o}(\mathcal{E}_\alpha)$ as

$$F_\alpha^{v,e}(w) = G_\alpha^{v,e}(\mathcal{E}_\alpha) \tag{26}$$

and

$$F_\alpha^{v,o}(w) = \begin{cases} G_\alpha^{v,o}(\mathcal{E}_\alpha), & w \ge 0 \\ -G_\alpha^{v,o}(\mathcal{E}_\alpha), & w \le 0. \end{cases} \tag{27}$$

Thus, in the wave frame, the complete equilibrium distribution functions are written in BGK form as

$$F^{v}_{\alpha}(w)=\begin{cases}G^{v,e}_{\alpha}(\mathcal{E}_{\alpha})+G^{v,o}_{\alpha}(\mathcal{E}_{\alpha}), & w\geq 0\\ G^{v,e}_{\alpha}(\mathcal{E}_{\alpha})-G^{v,o}_{\alpha}(\mathcal{E}_{\alpha}), & w\leq 0.\end{cases} \tag{28}$$

A. Even BGK functions $g^{v,e}_{\alpha}$

Following Refs. [12] and [13], a general representation for f^{e}_{α} is obtained by writing each $g^{v,e}_{\alpha}$ in Eq. (19) as $G^{v,e}_{\alpha}$ plus a variable function $h^{v,e}_{\alpha}$, so that

$$f^{e}_{\alpha}(\xi,w)=g^{v,e}_{\alpha}(\mathcal{E}_{\alpha})=G^{v,e}_{\alpha}(\mathcal{E}_{\alpha})+h^{v,e}_{\alpha}(\mathcal{E}_{\alpha}) \ . \tag{29}$$

When both $\varphi(\xi)$ and the functions $h^{v,e}_{\alpha}$ vanish, then $f^{e}_{\alpha}(\xi,w)=G^{e}_{\alpha}(\mathcal{E}_{\alpha})$, which again reduces exactly to the even part $F^{v,e}_{\alpha}(w)$ of the shifted equilibrium as in Eq. (26). Moreover, when both $\varphi(\xi)$ and the $h^{v,e}_{\alpha}$ are small, then by a Taylor expansion about $\varphi(\xi)=0$,

$$\begin{aligned}f^{e}_{\alpha}(\xi,w)&=g^{v,e}_{\alpha}(\tfrac{1}{2}m_{\alpha}w^{2}+q_{\alpha}\varphi(\xi))\\&=g^{v,e}_{\alpha}(\tfrac{1}{2}m_{\alpha}w^{2})+\frac{dg^{v,e}_{\alpha}}{d\mathcal{E}_{\alpha}}(\tfrac{1}{2}m_{\alpha}w^{2})q_{\alpha}\varphi(\xi)+o(\varphi)\\&=G^{v,e}_{\alpha}(\tfrac{1}{2}m_{\alpha}w^{2})+h^{v,e}_{\alpha}(\tfrac{1}{2}m_{\alpha}w^{2})+\frac{dg^{v,e}_{\alpha}}{d\mathcal{E}_{\alpha}}(\tfrac{1}{2}m_{\alpha}w^{2})q_{\alpha}\varphi(\xi)+o(\varphi)\\&=F^{v,e}_{\alpha}(w)+O(\varphi,h^{v,e}_{\alpha}) \ ;\end{aligned} \tag{30}$$

hence $f^{e}_{\alpha}(\xi,w)$ is close to $F^{v,e}_{\alpha}(w)$.

In the above definition we have ignored a minor technical issue: Eq. (24) actually defines $G^{v,e}_{\alpha}(\mathcal{E}_{\alpha})$ only for $\mathcal{E}_{\alpha}\geq 0$, whereas when $\varphi(\xi)\neq 0$ the energy $\mathcal{E}_{\alpha}$ can be negative. Hence the definition Eq. (29) is inadequate. It is necessary, therefore, to construct a smooth non-negative extension of the definition of Eq. (24) to negative values of its argument. Since $G^{v,e}_{\alpha}(\mathcal{E}_{\alpha})\geq 0$, which follows from the positivity of the F_{α}, and since $G^{v,e}_{\alpha}(0)=F^{v,e}_{\alpha}(0)=F_{\alpha}(v)$, which is nonzero except in extremely special cases, such an extension can almost always be found. The resulting extended function $\tilde{G}^{e}_{\alpha}(\mathcal{E}_{\alpha})$ will agree with the definition of Eq. (24) for $\mathcal{E}_{\alpha}\geq 0$ and will have at least half as many continuous derivatives as the functions $F_{\alpha}(u)$. The details of a very general extension of this form have been worked out and they apply directly to the case at hand [12]; hence, in all that follows this extended function will be used. However, for simplicity we shall denote the extended function by the same symbol $G^{v,e}_{\alpha}$. As discussed earlier, the substitution of $f^{e}_{\alpha}(\xi,w)=g^{v,e}_{\alpha}(\mathcal{E}_{\alpha})$ into Poisson's equation, Eq. (16), results in the ordinary differential equation for $\varphi(\xi)$, Eq. (21), which now clearly depends parametrically on the variable functions $h^{v,e}_{\alpha}$.

B. Odd BGK functions $g^{v,o}_{\alpha}$

In Sec. V we shall construct parametrized branches of solutions $\varphi(\xi)$ of Eq. (21) that take the form $(\varphi(\xi;\varphi_0),h^{v,e}_{\alpha}(\varphi_0))$, where the parameter φ_0 is closely related to the amplitude of the electric potential. On any such branch, a small value of φ_0 specifies an exact nonlinear solution of Eqs. (15) and (16), where $\varphi(\xi)$ is small and f^{e}_{α} is close to $F^{v,e}_{\alpha}(w)$. Furthermore, the branches of solutions we shall construct are continuously connected to the equilibrium in the sense that

$$\lim_{\varphi_0\to 0}(\varphi(\xi;\varphi_0),h^{v,e}_{\alpha}(\varphi_0))=(0,0) \ , \tag{31}$$

in which case f^{e}_{α} reduces exactly to $F^{v,e}_{\alpha}(w)$. For any such branch of solutions, it is possible to develop appropriate definitions for the BGK functions $g^{v,o}_{\alpha}$ that are used to represent the odd parts f^{o}_{α} as given in Eq. (20). The definition for $g^{v,o}_{\alpha}$ analogous to Eq. (29) for $g^{v,e}_{\alpha}$ is, using the function $G^{v,o}_{\alpha}$ defined in Eq. (25),

$$g^{v,o}_{\alpha}(\mathcal{E}_{\alpha})=G^{v,o}_{\alpha}(\mathcal{E}_{\alpha})+h^{v,o}_{\alpha}(\mathcal{E}_{\alpha}) \ , \tag{32}$$

where the $h^{v,o}_{\alpha}$ are another set of variable functions. This definition is in general unsatisfactory however, since Eq. (20) implies that $f^{o}_{\alpha}(\xi,w)$ must vanish along the line $w=0$. For if we approach this line from above we find $\lim_{w\to 0+}f^{o}_{\alpha}(\xi,w)=g^{v,o}_{\alpha}(q_{\alpha}\varphi(\xi))$, while from below we have in contrast $\lim_{w\to 0-}f^{o}_{\alpha}(\xi,w)=-g^{v,o}_{\alpha}(q_{\alpha}\varphi(\xi))$. This problem, associated again with particle trapping, necessitates smooth modification of the definition of Eq. (32) so that

$$g^{v,o}_{\alpha}(\mathcal{E}_{\alpha})=0, \quad \mathcal{E}_{\alpha}\leq Q_{\alpha}\equiv|q_{\alpha}\varphi|_{\max} \ . \tag{33}$$

In other words, the functions $h^{v,o}_{\alpha}$ must depend not only on $\mathcal{E}_{\alpha}$ but on the amplitude parameter φ_0 as well. Using $g^{v,o}_{\alpha}$ in Ampère's equation, Eq. (18), gives

$$0=8\pi\sum_{\alpha}q_{\alpha}\int_{0}^{\infty}dw\, w\, g^{v,o}_{\alpha}(\tfrac{1}{2}m_{\alpha}w^{2}+q_{\alpha}\varphi(\xi)) \ , \tag{34}$$

which is a zero current constraint in the wave frame. Definitions of the odd functions $g^{v,o}_{\alpha}$, that satisfy Eqs. (33) and (34), and are suitable for each of the cases considered in this paper have been developed in Ref. [13]. Because the details are technically rather complicated and would lead us away from our main line of development, we omit them here and instead summarize them in Appendix A. The important conclusion established there is that, given a solution $\varphi(\xi)$ to Eq. (21), the functions $g^{v,o}_{\alpha}$ always can be defined in such a way that (i) the zero-current constraint of Eq. (34) is satisfied, (ii) the overall distribution functions f_{α} are non-negative, and (iii) the odd parts f^{o}_{α}, given in Eq. (20), uniformly approach the odd parts of the velocity-shifted equilibrium $F_\alpha^{v,o}$ as the wave amplitude goes to zero. Thus, corresponding to any near-equilibrium solution of Eqs. (15) and (16) is at least one and usually many physically reasonable solutions of Eqs. (17) and (18).

V. MECHANICAL POTENTIAL AND NONLINEAR TRAVELING WAVE SOLUTIONS

We now exploit the representation scheme just developed to construct exact nonlinear traveling wave solutions to the full Vlasov-Poisson-Ampère equations. Recall that after shifting to a wave frame translating with velocity v, in which the position and velocity variables are $\xi = x - vt$ and $w = u - v$, and decomposing the wave frame distribution functions $f_\alpha(\xi,w)$ into their even and odd parts with respect to w, the Vlasov-Poisson-Ampère equations were seen to divide into two pairs of time-independent equations: Eqs. (15) and (16) for $f_\alpha^e(\xi,w)$ and $\varphi(\xi)$ and Eqs. (17) and (18) for $f_\alpha^o(\xi,w)$ and $\varphi(\xi)$. The first pair was reduced to one equation for $\varphi(\xi)$ by virtue of the convenient representation, Eq. (29), which automatically satisfied Eq. (15) and guaranteed that for small $\varphi(\xi)$ and $h_\alpha^{v,e}$, f_α^e was close to $F_\alpha^{v,e}(w)$. Poisson's equation thereby became a basic nonlinear differential equation for $\varphi(\xi)$,

$$-\frac{d^2\varphi}{d\xi^2} = 4\pi \sum_\alpha q_\alpha \int_{-\infty}^{\infty} dw\, g_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2 + q_\alpha \varphi(\xi)) \ . \quad (35)$$

Thus, through a judicious choice of a BGK representation for the plasma distribution functions, the entire problem of the existence of small but constant amplitude nonlinear waves near an equilibrium with distribution functions $F_\alpha(u)$ is reduced to that of solving the nonlinear differential equation given by Eq. (35). This equation contains a great deal of physics and in particular determines, as we shall see, the relationship between frequency ω and wave number k for spatially periodic BGK waves in the small amplitude limit. Holloway and Dorning [13] approached Eq. (35) as a bifurcation problem with the functions $h_\alpha^{v,e}$ considered as an infinite-dimensional set of variable parameters. Through Lyapunov-Schmidt reduction and subsequent bifurcation analysis, they were then able to show that if $\kappa^2(v) > 0$, where

$$\kappa^2(v) = 4\pi \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int dw \frac{1}{w} \frac{d}{dw} F_\alpha^{v,e}(w) \ , \quad (36)$$

there exist branches of small amplitude spatially periodic solutions $\varphi(\xi)$ of Eq. (35) that bifurcate from the zero solution as the $h_\alpha^{v,e}$ are increased from zero. Such solutions correspond to periodic traveling waves of velocity v in the original frame of reference. While their analysis is correct, there nevertheless exists a much simpler, more intuitive, and yet equally rigorous method for constructing these solutions, as we shall now show. We shall employ the BGK representation developed in Sec. IV and focus upon the fundamental nonlinear differential equation for $\varphi(\xi)$, Eq. (35). For clarity we shall first consider the simplest case in which the variable functions $h_\alpha^{v,e}$ are identically zero, in which case we have just $g_\alpha^{v,e} = G_\alpha^{v,e}$.

The key to the formulation is an observation by Bernstein, Greene, and Kruskal [11] that Eq. (35) may be cast in the convenient form

$$\frac{d^2\varphi}{d\xi^2} = -\frac{d}{d\varphi} A_v(\varphi) \ , \quad (37)$$

where the "mechanical potential" $A_v(\varphi)$ is introduced as

$$A_v(\varphi) \equiv 4\pi \sum_\alpha q_\alpha \int_0^\varphi d\varphi' \int_{-\infty}^{\infty} dw\, G_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2 + q_\alpha \varphi') \quad (38)$$

and is obtained by formally integrating the right-hand side of Eq. (35) once with respect to φ. The form of Eq. (37) is particularly convenient since, after multiplying by $d\varphi/d\xi$, it may be integrated once to obtain a first integral

$$\Lambda = \frac{1}{2}\left[\frac{d\varphi}{d\xi}\right]^2 + A_v(\varphi) \ . \quad (39)$$

This important exact integral Λ was not exploited in the previous analyses [12,13], even though its existence allows the problem to be solved completely without recourse to methods of nonlinear functional analysis. For Eq. (37) has, in fact, the simple form of Newton's equation for particle motion in a one-dimensional potential $A_v(\varphi)$: the integral Λ plays the role of the particle's energy and φ that of its position, while ξ takes the place of the independent time variable. Thus, by analogy with the one-dimensional motion of a classical particle in a potential well, the character of the solutions to Eq. (37) is determined entirely by the shape of $A_v(\varphi)$, which, in turn, depends both upon the phase velocity v and the distribution functions $F_\alpha(u)$ of the underlying plasma equilibrium, which enter into Eq. (37) through the definition of $G_\alpha^{v,e}$ in Eq. (38). Therefore, to find the solutions $\varphi(\xi)$, if any exist, which represent traveling waves of velocity v near an equilibrium F_α, we need only use the explicit definition of $G_\alpha^{v,e}$ to calculate the shape of the potential $A_v(\varphi)$.

Unfortunately, in general, the integrals in Eq. (38), though certainly amenable to numerical evaluation, cannot be performed analytically so as to yield a useful closed form expression for $A_v(\varphi)$. On the other hand, since our main interest is BGK waves of small amplitude, we need only know the shape of $A_v(\varphi)$ in the neighborhood of $\varphi=0$; hence we can proceed by expanding $A_v(\varphi)$ in powers of φ. Small amplitude waves are of particular importance, of course, not only because the existence of such waves reflects upon the nonlinear stability of the Vlasov equilibria but also because they play a physically important role in any collisionless plasma subject to weak perturbing influences. Considering equilibria for which the $F_\alpha(u)$ are sufficiently smooth, we have for small φ

$$A_v(\varphi) = \frac{1}{2} A_v^{(2)}\varphi^2 + \frac{1}{3!} A_v^{(3)}\varphi^3 + \frac{1}{4!} A_v^{(4)}\varphi^4 + o(\varphi^4) \ , \quad (40)$$

where

$$A_v^{(i)} \equiv \left[\frac{d^i}{d\varphi^i} A_v(\varphi) \right]_{\varphi=0} = 4\pi \sum_\alpha \frac{q_\alpha^i}{m_\alpha^{i-1}} \int_{-\infty}^{\infty} dw \left[\frac{1}{w} \frac{d}{dw} \right]^{i-1} F_\alpha^{v,e}(w) \ , \quad i=2,3,\ldots, \tag{41}$$

which follows from the definitions in Eqs. (24) and (38).

In Eq. (40) we have taken account of the fact that the coefficients of the first two terms in the expansion vanish: $A_v^{(0)} = A_v(0) = 0$ by virtue of the definition of $A_v(\varphi)$, Eq. (38), while by direct calculation

$$A_v^{(1)} = 4\pi \sum_\alpha q_\alpha \int dw \, F_\alpha^{v,e}(w) = 0 \ , \tag{42}$$

which vanishes since it is precisely the charge density resulting from the velocity shifted Vlasov equilibrium $F_\alpha(w+v)$. For purposes of analysis it is also convenient, using Eq. (36), to replace the coefficient $A_v^{(2)}$ in Eq. (40) with the exactly equivalent quantity $\kappa^2(v)$. With this change in notation Eq. (37) then takes, for small $\varphi(\xi)$, the form

$$\frac{d^2\varphi}{d\xi^2} = -\kappa^2(v)\varphi - \frac{1}{2} A_v^{(3)}\varphi^2 - \frac{1}{3!} A_v^{(4)}\varphi^3 + o(\varphi^3) \ . \tag{43}$$

It is our goal in this section to investigate methodically the small amplitude solutions $\varphi(\xi)$ of Eq. (43) as the coefficients $\kappa^2(v)$, $A_v^{(3)}$, and $A_v^{(4)}$ take various possible values. In most cases, $\kappa^2(v)$ and $A_v^{(3)}$ determine the shape of $A_v(\varphi)$ in the neighborhood of $\varphi=0$ and therefore the form of the small amplitude traveling wave solutions of velocity v. Exceptions will occur, however, when both $\kappa^2(v)$ and $A_v^{(3)}$ vanish simultaneously, i.e., at the same velocity v, in which case $A_v^{(4)}$ and possibly higher-order terms become important. Explicit examples of the above coefficients, exhibiting their dependence upon the phase velocity v, will be given in Sec. VI for an e^--p^+ thermal plasma; however, it is important to discuss them first in a more general context.

A. Spatially periodic waves

From Eq. (43) let us first rederive the sufficiency condition $\kappa^2(v) > 0$, originally derived in Refs. [12] and [13], for the existence of small but constant amplitude, spatially periodic traveling waves of velocity v arbitrarily close to an equilibrium $F_\alpha(u)$. We shall find that this condition results almost trivially in the context of the perspective based on the simple mechanical potential introduced above.

In the expansion of the mechanical potential $A_v(\varphi)$, the parameter $\kappa^2(v)$ ($=A_v^{(2)}$) is the lowest nonvanishing coefficient and gives the curvature of $A_v(\varphi)$ at $\varphi=0$. If $\kappa^2(v)>0$, then this curvature is positive, which implies the existence of a well of finite depth with its minimum at $\varphi=0$ as indicated schematically in Fig. 2(a). By analogy with single-particle motion in a confining well, there then exist solutions $\varphi(\xi)$ confined to this well that represent spatially periodic equilibria in the moving frame and spatially periodic traveling waves $\varphi(x-vt)$ when viewed in the laboratory frame of reference. Thus, at any velocity v for which $\kappa^2(v)>0$, there exists a family of spatially periodic traveling waves, parametrized by the mechanical energy $\Lambda = \frac{1}{2}(d\varphi/d\xi)^2 + A_v(\varphi)$, which includes waves of arbitrarily small but constant amplitude. Thus the sufficiency condition $\kappa^2(v)>0$ reported earlier in Ref. [13] has been obtained here quite simply. That the slightly less restrictive condition $\kappa^2(v) \geq 0$ is also sufficient follows from the fact that arbitrarily close to any equilibrium with $\kappa^2(v)=0$ is another with $\kappa^2(v)>0$ [12].

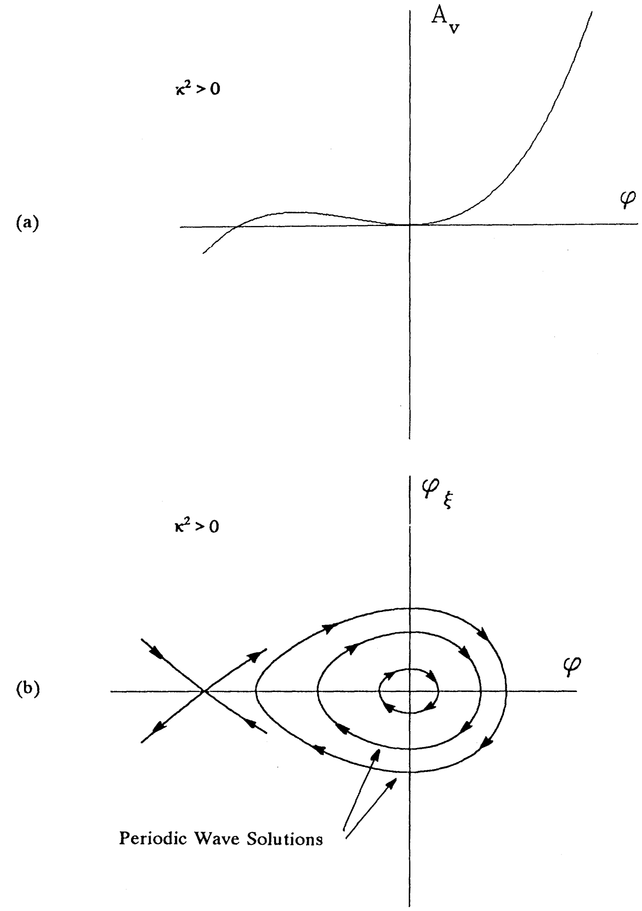

FIG. 2. (a) Whenever $\kappa^2(v)>0$ the mechanical potential $A_v(\varphi)$ contains a well of finite depth with minimum at $\varphi=0$. (b) Phase portrait in the (φ, φ_ξ) plane for the case $\kappa^2(v)>0$.

We can find the small amplitude form of these spatially periodic waves by keeping only the lowest-order term in $A_v(\varphi)$, in which case Eq. (43) becomes

$$\frac{d^2\varphi}{d\xi^2} + \kappa^2(v)\varphi = O(\varphi^2) \ . \tag{44}$$

Thus, in the small amplitude limit the solutions are sinusoidal with wave number $k=\kappa(v)$ ($=\sqrt{A_v^{(2)}}$) and frequency $\omega = \kappa(v)v$. Waves of small but finite amplitude are only approximately sinusoidal and can be expressed in terms of elliptic functions [by solving Eq. (43) with one more term retained]; thus they contain components at the harmonics of the basic wave number $\kappa(v)$. Corresponding to the translational invariance of Eq. (44) there is actually an infinite set of solutions having the same amplitude φ_0, distinguished only by the value of the boundary condition $\varphi(0)$. Physically, however, these waves differ from one another only by an inconsequential spatial shift.

It is informative to represent these spatially periodic traveling wave solutions as trajectories in an abstract two-dimensional phase space where $(\varphi,\varphi_\xi \equiv d\varphi/d\xi)$ are the phase variables and ξ plays the role of the independent timelike parameter. The phase flow is determined by Eq. (37), which can be rewritten as a pair of first-order equations

$$\frac{d}{d\xi}\varphi=\varphi_\xi,\quad \frac{d}{d\xi}\varphi_\xi=-\frac{d}{d\varphi}A_v(\varphi)\ . \tag{45}$$

The structure of the flow for the case $\kappa^2>0$ is shown in Fig. 2(b), where the stable fixed point located at $(\varphi,\varphi_\xi)=(0,0)$ corresponds to the velocity-shifted field-free Vlasov equilibrium $F_\alpha^v(w)=F_\alpha(v+w)$. Each of the closed periodic orbits surrounding this point corresponds to a spatially periodic equilibrium solution in the wave frame or to a spatially periodic traveling wave in the original frame. The perspective offered by viewing the solutions in the (φ,φ_ξ) phase plane will be particularly useful when considering aperiodic traveling wave solutions in Secs. V B–V D to follow.

The condition $k=\kappa(v)$ $(=\sqrt{A_v^{(2)}})$ leads to a dispersion relation for spatially periodic BGK waves in the small amplitude limit. Using the definition of $\kappa^2(v)$ in Eq. (36), we have

$$k^2=\kappa^2(\omega/k)=4\pi\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\int dw\frac{1}{w}\frac{d}{dw}F_\alpha^{v,e}(w)\ . \tag{46}$$

This expression can be written in a more familiar form by replacing $F_\alpha^{v,e}$ on the right-hand side with the full distribution functions F_α^v and introducing the principal value, in which case the odd parts $F_\alpha^{v,o}$ of the distribution functions do not contribute to the integrals. Upon reverting to the variable $u=v+w$ and recalling that $v=\omega/k$, we then find

$$1-\frac{4\pi}{k^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\mathrm{P}\int du\frac{F'_\alpha(u)}{u-\omega/k}=0\ , \tag{47}$$

which is the dispersion relation obtained by Vlasov, Eq. (13). Thus we conclude that the Vlasov dispersion relation does indeed apply—Vlasov's somewhat *ad hoc* development notwithstanding—to an important class of plasma waves. Equation (47) correctly gives the relationship between ω and k for spatially periodic BGK waves in the limit of small amplitude.

In the preceding discussion we considered the simple case $g_\alpha^{v,e}=G_\alpha^{v,e}$ in which the variable functions $h_\alpha^{v,e}$ were taken to be identically zero. More generally, however, we can write $g_\alpha^{v,e}=G_\alpha^{v,e}+h_\alpha^{v,e}$. This more general representation scheme presents no obstacles to the mechanical potential formalism, the mechanical potential merely depends now on the functions $h_\alpha^{v,e}$ as well as on the phase velocity v and thus we write it as $A_{v,h}(\varphi)$, where h denotes the entire set of functions $h_\alpha^{v,e}$. To consider small amplitude solutions we again expand $A_{v,h}(\varphi)$, now obtaining

$$\begin{aligned}A_{v,h}(\varphi)=&\,\delta A^{(1)}\varphi+\tfrac{1}{2}[\kappa^2(v)+\delta A^{(2)}]\varphi^2\\&+\frac{1}{3!}(A_v^{(3)}+\delta A^{(3)})\varphi^3\\&+\frac{1}{4!}(A_v^{(4)}+\delta A^{(4)})\varphi^4+o(\varphi^4)\ ,\end{aligned} \tag{48}$$

where the $\delta A^{(i)}$ are given by

$$\delta A^{(i)}=4\pi\sum_\alpha q_\alpha^i\int dw\frac{d^{(i-1)}h_\alpha^{v,e}}{d\mathcal{E}_\alpha^{(i-1)}}(\tfrac{1}{2}m_\alpha w^2)\ . \tag{49}$$

Using this mechanical potential in Eq. (37) then gives the differential equation

$$\begin{aligned}\frac{d^2\varphi}{d\xi^2}=&-\delta A^{(1)}-[\kappa^2(v)+\delta A^{(2)}]\varphi-\frac{1}{2}(A_v^{(3)}+\delta A^{(3)})\varphi^2\\&-\frac{1}{3!}(A_v^{(4)}+\delta A^{(4)})\varphi^3+o(\varphi^3)\ .\end{aligned} \tag{50}$$

If the above coefficients $\delta A^{(i)}$ are made arbitrarily small by an appropriate choice of functions $h_\alpha^{v,e}$, then when $\kappa^2(v)>0$ a simple calculation shows that $A_{v,h}(\varphi)$ now has a local minimum at

$$\begin{aligned}\varphi_{\min}=&-(1/\kappa^2)\delta A^{(1)}-(A_v^{(3)}/2\kappa^6)(\delta A^{(1)})^2\\&+(1/\kappa^4)\delta A^{(1)}\delta A^{(2)}+O(\delta A^3)\ ,\end{aligned} \tag{51}$$

where δA^3 signifies quantities that are third order in the coefficients $\delta A^{(i)}$. The curvature of $A_{v,h}(\varphi)$ at the location of this minimum is

$$\left.\frac{d^2A_{v,h}}{d\varphi^2}\right|_{\varphi_{\min}}=\kappa^2(v)+\delta A^{(2)}-\frac{A_v^{(3)}}{\kappa^2(v)}\delta A^{(1)}+O(\delta A^2)\ , \tag{52}$$

which is positive when the $\delta A^{(i)}$ are sufficiently small. Therefore, there exists a well of finite depth centered at $\varphi_{\min}$ and, as before, periodic trajectories $\varphi(\xi)$ trapped in this well correspond to traveling waves of velocity v, although for these solutions the electric potential is centered about the nonzero value $\varphi_{\min}$. For each choice of the small functions $h_\alpha^{v,e}$ (provided that the $\delta A^{(i)}$ are also small) we thus obtain a set of small amplitude periodic traveling wave solutions of velocity v, which may be labeled by the mechanical energy $\Lambda_h=\frac{1}{2}(d\varphi/d\xi)^2+A_{v,h}(\varphi)$. When $h_\alpha^{v,e}=0$ we of course recover the solutions discussed previously.

The variability of the functions $h_\alpha^{v,e}$ allows the construction of branches of small amplitude solutions of diverse properties. For instance, if the $h_\alpha^{v,e}$ are increased continuously from zero, then for each set of $h_\alpha^{v,e}$ the mechanical potential has a certain specific form and one may choose the energy Λ_h, or equivalently the amplitude φ_0, in order to select a definite solution $\varphi(\xi)$. Choosing φ_0 carefully in correspondence with the functions $h_\alpha^{v,e}$, i.e., $\varphi_0=\varphi_0(h_\alpha^{v,e})$, yields a parametrized branch of solutions that are continuously connected to the equilibrium. That is, $\varphi(\xi)\to 0$ and $f_\alpha(\xi,w)\to F_\alpha^v(w)$ as $h_\alpha^{v,e}\to 0$. Thus the flexibility introduced by the variable functions $h_\alpha^{v,e}$ leads to the existence of a broad class of small amplitude spatially periodic wave solutions of velocity v near the equilibrium $F_\alpha(u)$. Holloway and Dorning [13,15] have used a procedure similar to that just described to construct small amplitude plasma waves with specific tailored properties.

As an important illustrative example of these methods, we construct in Appendix B a branch of wave solutions for which the space average of the particle density for each species is the same as that in the underlying equilibrium. If a plasma equilibrium is subjected to a perturbing influence that does not introduce particles into the plasma or remove particles from it, such as an externally applied electrostatic field, then the plasma must at all times contain the same number of particles of each species (or same space-average particle density for an unbounded plasma) as it did initially. Neither the constant amplitude, spatially periodic traveling waves reported previously in Refs. [12] and [13] nor those discussed earlier in this section have this property; thus such waves cannot be excited by a plasma perturbation that leaves the number of particles unaltered. Since plasmas in the laboratory and in many natural environments are subjected predominantly to perturbing influences of this type, it is perhaps appropriate to question the physical relevance of these nonlinear wave solutions. However, similar waves for which the particle number density is the same as in the underlying equilibrium do exist, as we demonstrate explicitly in Appendix B. These waves are clearly physically relevant and the analysis through which they are obtained in Appendix B is a special case of the general analysis outlined above.

Before proceeding to describe small amplitude BGK waves that are not spatially periodic, such as solitary waves for instance, we shall make a few further comments concerning the relationship of these periodic waves to those familiar from the linear theory. Not surprisingly, the function $\kappa^2(v)$, which plays such a critical role for the nonlinear waves, appears also in the Landau dispersion relation $D_F(\omega,k)=0$. Written so as to explicitly include $\kappa^2(v)$, for the case of real ω (no damping), this relation becomes

$$\begin{aligned} D_F(\omega,k) &= 1-\frac{4\pi}{k^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \mathrm{P}\int du \frac{F'_\alpha(u)}{u-\omega/k} \\ &\quad -i\frac{\pi}{k^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}F'_\alpha\left[\frac{\omega}{k}\right] \\ &= 1-\frac{\kappa^2(v)}{k^2}-i\frac{\pi}{k^2}\sum_\alpha \frac{q_\alpha^2}{m_\alpha}F'_\alpha\left[\frac{\omega}{k}\right]=0\ . \end{aligned} \tag{53}$$

Setting the real and imaginary parts of this expression separately equal to zero yields two independent conditions

$$1-\kappa^2(v)/k^2=0 \tag{54}$$

and

$$F'_\alpha(\omega/k)=0\ . \tag{55}$$

The first of these relations requires $\kappa^2(v)=k^2>0$, which shows that the necessity (though not the sufficiency) of $\kappa^2(v)>0$ for small amplitude undamped waves follows already from the linear analysis. The second condition $F'_\alpha(\omega/k)=0$ indicates that within the linear description undamped waves are possible only if the equilibrium distribution functions $F_\alpha(u)$ have the rather special property that their velocity derivatives vanish at the phase velocity of the wave. The solutions of the linear theory in this case have been discussed in some detail by Van Kampen [8] and Case [9] and correspond to roots of the Landau dispersion relation that are imbedded in the continuous spectrum of the linearized operator (which occupies the imaginary axis). Together Eqs. (54) and (55) are sufficient for the existence of undamped waves according to the linear theory.

Conversely, if Eq. (55) is not satisfied for real ω, then ω must be complex, which leads to the exponentially damped or growing waves most familiar in the linear theory. For waves that are weakly damped, however, i.e., $|\mathrm{Im}(\omega)/\mathrm{Re}(\omega)|\ll 1$, it is known [2] that the linear theory yields the Vlasov dispersion relation as the correct relationship between $\mathrm{Re}(\omega)$ and k. Since the Vlasov dispersion relation also describes nonlinear spatially periodic BGK waves in the small amplitude limit, there thus exists a general correspondence between the weakly damped waves of the linear theory and small but constant amplitude, spatially periodic BGK waves. Namely, *to every weakly damped wave solution of the linear theory there corresponds an exact nonlinear and undamped spatially periodic BGK wave that has, in the small amplitude limit, the same frequency and wave number.*

B. Small amplitude solitary waves

We shall now extend the preceding results on small but constant amplitude spatially periodic waves to include small amplitude traveling solitary waves, as well as traveling double layers, sometimes called "kinks," which are moving transition regions between parts of the plasma that exist at different constant values of the electric potential. The mechanical potential formalism proves very convenient for this task; the solutions are discovered quite naturally by considering the general changes that occur in the shape of $A_v(\varphi)$ as the important parameter $\kappa^2(v)$ passes through zero. For any plasma equilibrium with distribution functions $F_\alpha(u)$, we shall define a *plasma critical velocity* as any velocity v_c at which the parameter $\kappa^2(v)$ vanishes, i.e., $\kappa^2(v_c)=0$. In Sec. VII we shall demonstrate that virtually any physically relevant plasma equilibrium will possess at least one such critical velocity. Typically, at $v=v_c$, both the derivative $(d\kappa^2/dv)(v_c)$ and the parameter $A_{v_c}^{(3)}$ will be nonzero. For definiteness, we consider the case in which $(d\kappa^2/dv)(v_c)>0$ and $A_{v_c}^{(3)}>0$, as shown schematically in the inset in Fig. 3(a).

We first examine the case in which the variable functions $h_\alpha^{v,e}$ of the BGK representation $f_\alpha^e(\xi,w)=G_\alpha^{v,e}(\mathcal{E}_\alpha)+h_\alpha^{v,e}(\mathcal{E}_\alpha)$ are identically zero, so that $f_\alpha^e(\xi,w)=G_\alpha^{v,e}(\mathcal{E}_\alpha)$ and the definition of the mechanical

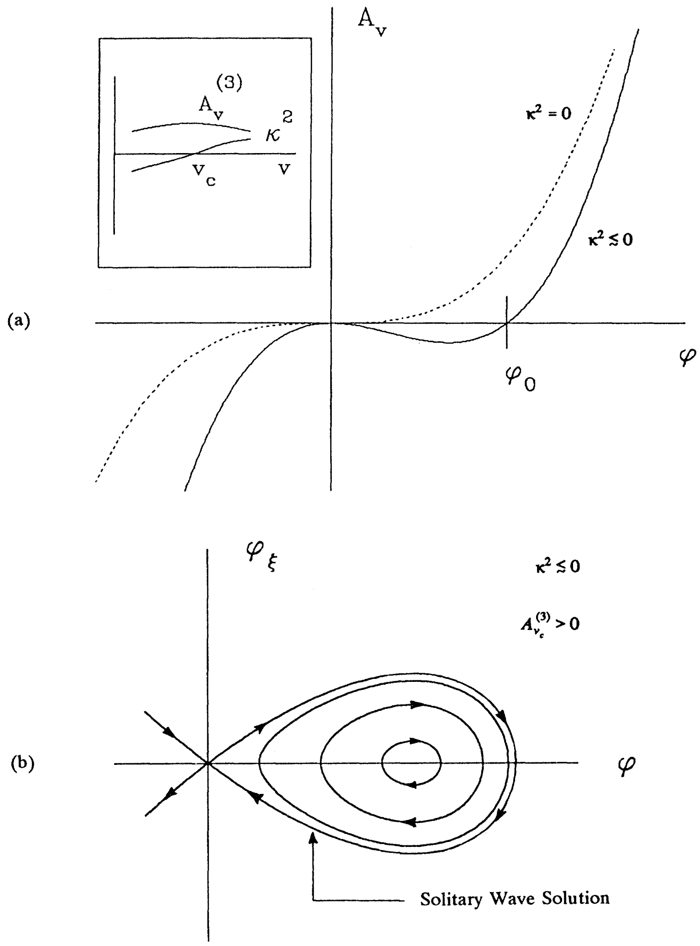

FIG. 3. (a) Local form of the mechanical potential $A_v(\varphi)$ for $v = v_c$ (dashed curve) and for $v \lesssim v_c$ (solid curve), where v_c is a critical velocity satisfying $\kappa^2(v_c)=0$. As suggested in the inset, the case considered here is $A_{v_c}^{(3)}>0$. The curves for $A_{v_c}^{(3)}<0$ are obtained from those indicated by reflection about the A_v axis. (b) Phase portrait in the (φ,φ_ξ) plane for the case $\kappa^2(v)\lesssim 0$.

potential is that given by Eq. (40). We shall demonstrate that a small well always exists in $A_v(\varphi)$ as $v \rightarrow v_c$ from below and, furthermore, that one of the trajectories $\varphi(\xi)$ for "motion" in this well corresponds to a traveling solitary wave. To see this, first consider the shape of $A_v(\varphi)$ in the neighborhood of $\varphi=0$ when $v=v_c$. Since we are considering the case in which $\kappa^2(v_c)=0$ and $A_{v_c}^{(3)}>0$, the mechanical potential has the form

$$A_{v_c}(\varphi)=\frac{1}{3!}A_{v_c}^{(3)}\varphi^3+O(\varphi^4) \tag{56}$$

in which the lowest-order term is cubic. Thus $A_{v_c}(\varphi)$ has the local form indicated by the dashed curve in Fig. 3(a). For $v \lesssim v_c$ on the other hand, $A_v(\varphi)$ contains, in addition to the cubic term, a quadratic term with small negative coefficient since $\kappa^2(v)<0$ but small for $v \lesssim v_c$. Clearly, this weak negative quadratic term modifies the dashed curve of Fig. 3(a) into the solid curve, where a small well has formed to the right of $\varphi=0$. The turning point φ_0 certainly exists for v sufficiently close to v_c since $\kappa^2(v)$, the coefficient of the quadratic term, goes to zero as $v \rightarrow v_c$. The formation of this well for v close to v_c depends only upon the condition $A_{v_c}^{(3)}\neq 0$, though to which side of $\varphi=0$ it forms depends on the sign of $A_{v_c}^{(3)}$.

Now with each trajectory $\varphi(\xi)$ in the mechanical potential $A_v(\varphi)$ we may associate, from Eq. (39), a conserved mechanical energy $\Lambda=\frac{1}{2}(d\varphi/d\xi)^2+A_v(\varphi)$. For the $\Lambda=0$ separatrix trajectory in the well with turning point φ_0, this energy expression gives the equation

$$\frac{1}{2}\left[\frac{d\varphi}{d\xi}\right]^2-\frac{1}{2}|\kappa^2(v)|\varphi^2+\frac{1}{3!}A_v^{(3)}\varphi^3+O(\varphi^4)=0\ , \tag{57}$$

where $|\kappa^2(v)|$ is small. Upon rearrangement and omission of the fourth- and higher-order terms, Eq. (57) becomes

$$\frac{d\varphi}{d\xi}=\pm\varphi(\xi)\sqrt{|\kappa^2(v)|-\tfrac{1}{3}A_v^{(3)}\varphi(\xi)}\ . \tag{58}$$

Integrating this equation once then gives the solution of Eq. (57) to lowest order in κ^2, which when shifted back into the original frame of reference by making the replacement $\xi=x-vt$ becomes the solitary wave

$$\varphi(x-vt)=\varphi_0(v)\text{sech}^2[\tfrac{1}{2}|\kappa(v)|(x-vt)]\ ,\qquad \varphi_0(v)=-3\frac{\kappa^2(v)}{A_{v_c}^{(3)}}\ . \tag{59}$$

Thus the above discussion demonstrates that a branch of traveling solitary waves, parametrized by the phase velocity v, bifurcates from an equilibrium $F_\alpha(u)$ as v passes through any critical velocity v_c that is a root of $\kappa^2(v)=0$. The condition $A_{v_c}^{(3)}\neq 0$ is also required and is satisfied except in special cases. The bifurcating solutions are reflected in Fig. 3(b) which shows the phase flow near the origin in the (φ,φ_ξ) plane for $\kappa^2 \lesssim 0$ and $A_{v_c}^{(3)}>0$. In this case the saddle point at $(\varphi,\varphi_\xi)=(0,0)$ corresponds to the velocity-shifted equilibrium $F_\alpha(w+v)$ and the homoclinic orbit corresponds to the traveling solitary wave solution that is described by Eq. (59) in the small amplitude limit. These solutions satisfy $v \gtrless v_c$ corresponding to $d\kappa^2/dv|_{v_c} \lessgtr 0$, since they exist for $\kappa^2<0$ and sufficiently small.

The even parts (with respect to the velocity of the wave) of the distribution functions corresponding to these solitary wave solutions are

$$\begin{aligned} f_\alpha^e(\xi,w) &= G_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2+q_\alpha\varphi(\xi)) \\ &= F_\alpha^{v,e}(w)+\frac{1}{w}\frac{dF_\alpha^{v,e}}{dw}(w)\frac{q_\alpha}{m_\alpha}\varphi(\xi)+O(\varphi^2) . \end{aligned} \tag{60}$$

As for the periodic waves, suitable corresponding odd parts of the distribution functions can be constructed using the BGK functions $g_\alpha^{v,o}$ defined in Appendix A and setting $h_\alpha^{v,e}=0$ (or, equivalently, $\mu_\alpha=0$ in Appendix A). Equation (59) gives the form of these solitary waves only in the small amplitude limit; finite amplitude waves may be expressed in terms of elliptic functions that reduce to Eq. (59) in the limit of small φ_0. The sech^2 form of these small amplitude solitary wave suggests a possible connection with the Korteweg–de Vries (KdV) equation, and such a connection has indeed been previously demonstrated for processes characterized by weak nonlinearity and weak dispersion [16,17]. If the condition $A_{v_c}^{(3)}\neq 0$ is not satisfied, then the bifurcating small amplitude waves take a different form, to be discussed later in this section, since then the coefficient $A_{v_c}^{(4)}$ is important in determining the small φ form of $A_v(\varphi)$ when $\kappa^2(v)$ is small.

The foregoing discussion applies to the simplest case $h_\alpha^{v,e}=0$, for which there is only a single branch of bifurcating solitary waves associated with each critical velocity v_c. However, with each v_c there is associated not just one but in fact a continuous infinity of similar but distinct bifurcating branches of solitary wave solutions. To show this, we now choose the functions $h_\alpha^{v,e}$ in the particular form $h_\alpha^{v,e}=\delta_\alpha r_\alpha^{v,e}$, where the δ_α are variable real parameters and the $r_\alpha^{v,e}$ are fixed functions. The BGK functions $g_\alpha^{v,e}=G_\alpha^{v,e}$ are thus modified to $\tilde{g}_\alpha^{v,e}=G_\alpha^{v,e}+\delta_\alpha r_\alpha^{v,e}$. Let the $r_\alpha^{v,e}$ also depend continuously on the phase velocity v in such a way that $r_\alpha^{v,e}|_{v=v_c}=0$. With these additional functions the important parameters $\kappa^2(v)$ and $A_v^{(3)}$ become

$$\tilde{\kappa}^2(v)=\kappa^2(v)+4\pi\sum_\alpha \frac{\delta_\alpha q_\alpha^2}{m_\alpha}\int dw\frac{1}{w}\frac{d}{dw}r_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2) , \tag{61}$$

$$\tilde{A}_v^{(3)}=A_v^{(3)}+4\pi\sum_\alpha \frac{\delta_\alpha q_\alpha^3}{m_\alpha^2}\int dw\frac{1}{w}\frac{d}{dw}\left[\frac{1}{w}\frac{d}{dw}r_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2)\right] , \tag{62}$$

where $\tilde{\kappa}^2(v_c)=\kappa^2(v_c)$ and $\tilde{A}_{v_c}^{(3)}=A_{v_c}^{(3)}$ since $r_\alpha^{v,e}$ has been chosen to vanish (i.e., become the zero function) at $v=v_c$.

Now by choosing each of the constants δ_α very small, we ensure that $\tilde{\kappa}^2(v)$ and $\tilde{A}_v^{(3)}$ differ only slightly from $\kappa^2(v)$ and $A_v^{(3)}$ at any particular value of v. In this case the arguments made previously still hold and demonstrate the existence of a small well in the mechanical potential in the neighborhood of $\varphi=0$ for v sufficiently close to v_c, and an associated bifurcating branch of solitary waves as v passes through v_c. Thus any set of small functions $r_\alpha^{v,e}$ satisfying the above properties leads to a bifurcating branch of solitary waves for which the electric potential and the even parts of the distribution functions are given in the small amplitude limit by Eqs. (58) and (59) with $\tilde{\kappa}^2(v)$ and $\tilde{A}_v^{(3)}$ replacing $\kappa^2(v)$ and $A_v^{(3)}$. That there is an uncountably infinite set of such branches follows immediately from the great freedom we have in choosing the functions $r_\alpha^{v,e}$.

C. Small amplitude traveling double layers

Plasma equilibria typically satisfy the condition $A_{v_c}^{(3)}\neq 0$. It is possible, however, that $A_{v_c}^{(3)}=0$, in which case the bifurcating small amplitude solutions take a slightly different form. An illustrative and physically relevant example occurs when $A_v(\varphi)$ is strictly an even function of φ, a feature obtained, for instance, via the charge-conjugation symmetry of a two-species plasma in which the particles have identical mass but opposite charge (e.g., an electron-positron plasma). In this case, since in general $A_{v_c}^{(4)}\neq 0$, the mechanical potential in the neighborhood of $\varphi=0$ changes, as v passes through v_c, between the dashed and solid shapes shown in Figs. 4(a) and 5(a) for $A_v^{(4)}>0$ and $A_v^{(4)}<0$, respectively. To describe the changes that occur as v passes through v_c it is useful to consider the fixed points of the dynamical system described by Eq. (45), which are determined by the conditions $\varphi_\xi=0$ and $dA_v(\varphi)/d\varphi=0$. For small φ and $v\simeq v_c$, and when $A_v(\varphi)=A_v(-\varphi)$, the latter condition is

$$\frac{dA_v(\varphi)}{d\varphi}=\varphi\left[\kappa^2+\frac{1}{6}A_{v_c}^{(4)}\varphi^2+o(\varphi^2)\right]=0 . \tag{63}$$

Neglecting the term $o(\varphi^2)$ and solving for φ gives the three fixed-point branches as functions of $\kappa^2(v)$,

$$\begin{aligned} &\varphi^{(0)}=0 , \\ &\varphi^{(\pm)}=\pm\varphi_0\equiv\pm\sqrt{-6\kappa^2(v)/A_{v_c}^{(4)}}+o(\kappa(v)) . \end{aligned} \tag{64}$$

In either case $A_v^{(4)}>0$ or $A_v^{(4)}<0$, as v passes through v_c and $\kappa^2(v)$ passes through zero, one fixed point suddenly becomes three, or vice versa, in a pitchfork bifurcation. The fixed points are of course associated with the local

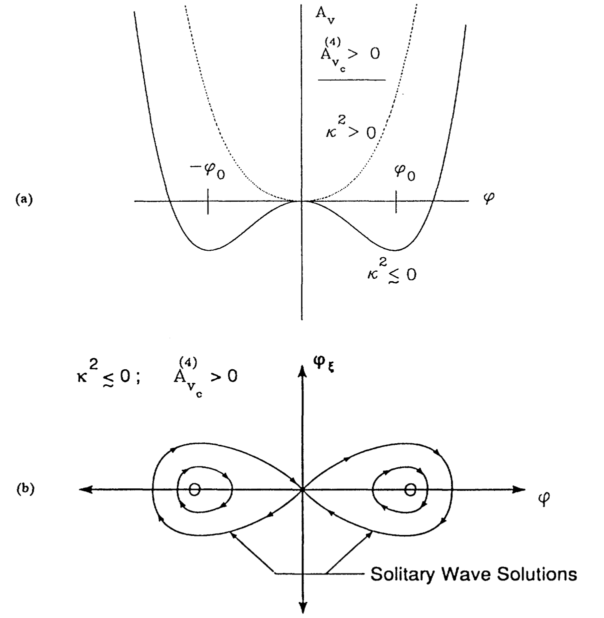

FIG. 4. Nongeneric case ($A_{v_c}^{(3)}=0$) with $A_{v_c}^{(4)}>0$. In (a) the local shape of $A_v(\varphi)$ is shown for $\kappa^2>0$ (dashed curve) and for $\kappa^2\lesssim 0$ (solid curve) where two symmetric wells have formed. In the latter case the phase portrait in the (φ,φ_ξ) plane has the form shown in (b), where the trajectories corresponding to solitary waves are labeled.

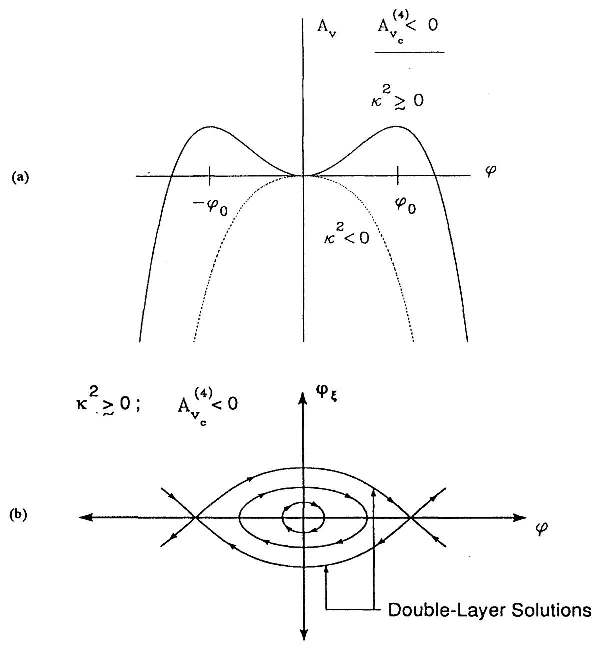

FIG. 5. Nongeneric case ($A_{v_c}^{(3)}=0$) with $A_{v_c}^{(4)}<0$. In (a) the local shape of $A_v(\varphi)$ is shown for $\kappa^2<0$ (dashed curve) and for $\kappa^2\gtrsim 0$ (solid curve) where two symmetric maxima have formed. In the latter case the phase portrait in the (φ,φ_ξ) plane has the form shown in (b), where the trajectories corresponding to double-layer waves are labeled.

maxima or minima of the mechanical potentials in Figs. 4(a) and 5(a).

The small amplitude solutions of Eq. (45) take one of two possible forms depending upon the sign of $A_{v_c}^{(4)}$. If $A_{v_c}^{(4)}>0$, then the solutions are quite similar to the solitary waves obtained previously for $A_{v_c}^{(3)}\neq 0$. The two symmetrically located wells in Fig. 4(a) exist for $\kappa^2\lesssim 0$ and the trajectories that begin (as $\xi\to-\infty$) and end (as $\xi\to+\infty$) upon the central local maximum correspond to solitary waves. The electric potential corresponding to these waves is found from the expression for the conserved quantity $\Lambda=\frac{1}{2}(d\varphi/d\xi)^2+A_v(\varphi)$, which, in the particular case $\Lambda=0$, is

$$\frac{1}{2}\left[\frac{d\varphi}{d\xi}\right]^2-\frac{1}{2}|\kappa^2(v)|\varphi^2+\frac{1}{4!}A_v^{(4)}\varphi^4+O(\varphi^5)=0\ , \tag{65}$$

where $|\kappa^2(v)|$ is small. This leads to the approximate equation

$$\frac{d\varphi}{d\xi}=\pm\varphi(\xi)\sqrt{|\kappa^2(v)|-\tfrac{1}{12}A_v^{(4)}\varphi^2(\xi)}\ , \tag{66}$$

which can be easily integrated. The resulting solitary wave electric potential in the laboratory frame is, in the small amplitude limit,

$$\begin{aligned}\varphi(x-vt)&=\pm\varphi_0\operatorname{sech}[|\kappa(v)|(x-vt)]\ ,\\ \varphi_0&=\left[-6\frac{\kappa^2(v)}{A_{v_c}^{(4)}}\right]^{1/2}\ .\end{aligned} \tag{67}$$

One obtains elliptic integrals, of course, if more terms are retained in Eq. (65). The phase flow on the three-branch side of the bifurcation is shown for this case in Fig. 4(b), where the two labeled homoclinic orbits correspond to solitary waves of opposite sign, the small amplitude limits of which are given by Eq. (67).

If, on the other hand, $A_{v_c}^{(4)}<0$, then the bifurcating solutions have the appreciably different form of translating double-layer waves (or kinks, as they are known in the context of nonlinear wave equations) in which the electric potential makes a smooth transition from one value to another while remaining essentially constant outside the transition region. Referring to the solid curve in Fig. 5(a), these solutions are generated, through an analogy with single-particle motion, by the trajectories that start atop either of the local maxima and end atop the other. These local maxima are ensured to exist for $\kappa^2(v)$ positive but sufficiently small, in which case the small amplitude limiting form of these waves is, in the laboratory frame,

$$\varphi(x-vt)=\pm\varphi_0\tanh\left[\frac{\kappa(v)}{\sqrt{2}}(x-vt)\right]\ , \tag{68}$$

where φ_0 is the same as in Eq. (67).

As for the spatially periodic waves and solitary waves, the variable functions $h_\alpha^{v,e}$ can be exploited to demonstrate that there exists not one but a continuous infinity of these branches of undamped traveling double-layer waves or nongeneric ($A_{v_c}^{(3)}=0$) solitary waves. These solutions can also be connected, in the context of weakly dispersive nonlinear processes, to a nonlinear wave equation, although in this case the relevant equation is the modified KdV equation [17].

D. Generic and nongeneric bifurcations

As discussed in Secs. V A and V B, the analysis of the nonlinear Poisson equation, Eq. (37), using the simple mechanical potential formulation, leads directly to spatially periodic, traveling wave solutions as well as solitary wave solutions that are of small but constant amplitude. These solutions are connected through a formal generic ($A_{v_c}^{(3)} \neq 0$) transcritical bifurcation, which occurs when $\kappa^2(v)$ decreases through zero, as v passes through a critical velocity v_c. In this bifurcation the center at $(\varphi,\varphi_\xi)=(0,0)$, which is associated with the Vlasov equilibrium and is surrounded by small amplitude periodic wave solutions, becomes a saddle point with a homoclinic orbit corresponding to a small amplitude solitary wave solution. A summary of this transition is depicted schematically in the bifurcation diagram of Fig. 6, which exhibits the trajectories in the two-dimensional phase space (φ,φ_ξ) as a function of the bifurcation parameter $\kappa^2(v)$.

Analogously, the related analysis of the nongeneric case ($A_{v_c}^{(3)}=0$) of a charge-symmetric plasma described in Sec. V C leads for $A_{v_c}^{(4)}>0$ to a transition from a center

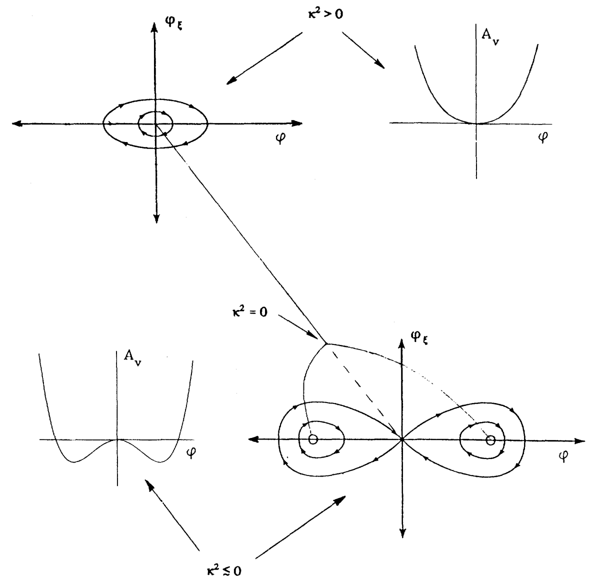

FIG. 7. Phase portraits in the (φ,φ_ξ) plane for $\kappa^2 \gtrsim 0$ and for $\kappa^2 \lesssim 0$ in the nongeneric case $A_{v_c}^{(3)}=0$, $A_{v_c}^{(4)}>0$. The single fixed-point branch $(\varphi,\varphi_\xi)=(0,0)$, which exists for $\kappa^2(v)>0$, meets the three fixed-point branches $(\varphi,\varphi_\xi)=(0,0)$ and $(\varphi,\varphi_\xi)=(\pm\sqrt{-6\kappa^2(v)/A_{v_c}^{(4)}},0)$, which exist for $\kappa^2(v)<0$, in a supercritical pitchfork bifurcation as $\kappa^2(v)$ passes through zero, i.e., as v passes through a nongeneric critical velocity v_c.

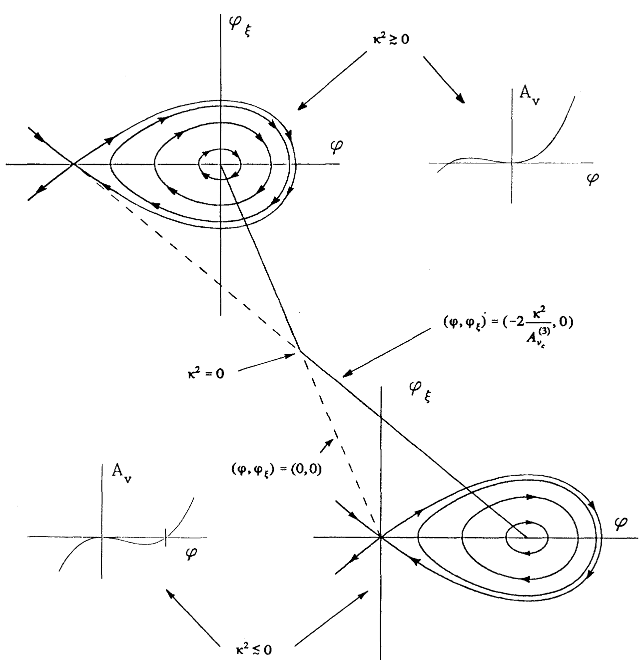

FIG. 6. Phase portraits in the (φ,φ_ξ) plane for $\kappa^2 \gtrsim 0$ and for $\kappa^2 \lesssim 0$ in the generic case $A_{v_c}^{(3)} \neq 0$. The fixed-point branches $(\varphi,\varphi_\xi)=(0,0)$ and $(\varphi,\varphi_\xi)=(-2\kappa^2(v)/A_{v_c}^{(3)},0)$ intersect and exchange stability in a transcritical bifurcation as $\kappa^2(v)$ passes through zero, i.e., as v passes through a generic critical velocity v_c.

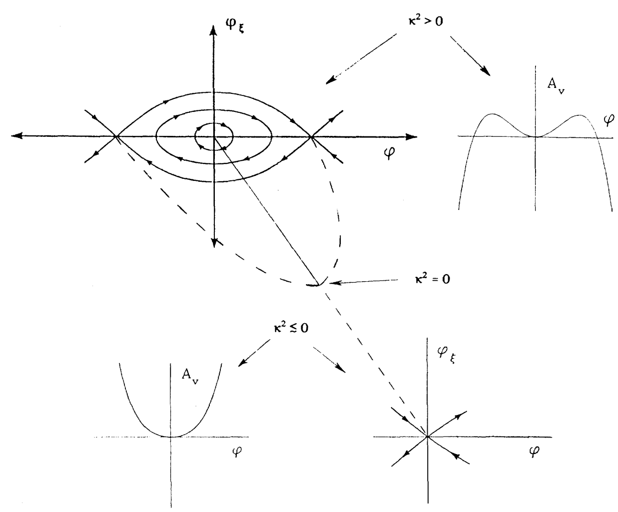

FIG. 8. Phase portraits in the (φ,φ_ξ) plane for $\kappa^2 \gtrsim 0$ and for $\kappa^2 \lesssim 0$ in the nongeneric case $A_{v_c}^{(3)}=0$, $A_{v_c}^{(4)}<0$. The single fixed-point branch $(\varphi,\varphi_\xi)=(0,0)$, which exists for $\kappa^2(v)<0$, meets the three fixed-point branches $(\varphi,\varphi_\xi)=(0,0)$ and $(\varphi,\varphi_\xi)=(\pm\sqrt{-6\kappa^2(v)/A_{v_c}^{(4)}},0)$, which exist for $\kappa^2(v)>0$, in a subcritical pitchfork bifurcation as $\kappa^2(v)$ passes through zero, i.e., as v passes through a nongeneric critical velocity v_c.

at $(\varphi,\varphi_\xi)=(0,0)$ surrounded by small amplitude periodic wave solutions for $\kappa^2(v)>0$ to a saddle point with two symmetric homoclinic orbits corresponding to a pair of solitary wave solutions of opposite sign. Again, the bifurcation parameter is $\kappa^2(v)$. This transition is a supercritical pitchfork bifurcation, as shown in the bifurcation diagram of Fig. 7. Finally, in the nongeneric case ($A_{v_c}^{(3)}=0$) when $A_{v_c}^{(4)}<0$, also discussed in Sec. V C, the analysis shows that the transition from the center to the saddle point is a subcritical pitchfork bifurcation. Hence, as indicated in the bifurcation diagram of Fig. 8, the two heteroclinic orbits corresponding to a pair of small amplitude traveling double-layer or kink solutions of opposite sign exist for $\kappa(v)>0$, as do the small amplitude periodic waves. Figures 6–8 qualitatively summarize the general form of the main results obtained here using the simple mechanical potential formulation, although they of course do not indicate any of the complicated details involved in the construction of the related particle distribution functions that were described above.

VI. EXAMPLE: e^--p^+ THERMAL PLASMA

We shall now illustrate the foregoing developments explicitly in an important practical case: a thermal electron-proton plasma. In this case the even parts of the velocity-shifted distribution functions are

$$F_\alpha^{v,e}(w)=\frac{n_0}{2}\left[\frac{m_\alpha}{\pi k T_\alpha}\right]^{1/2}\times\left\{\exp\left[-\frac{m_\alpha}{2kT_\alpha}(w+v)^2\right]+\exp\left[-\frac{m_\alpha}{2kT_\alpha}(-w+v)^2\right]\right\}, \quad (69)$$

where $\alpha=e$ and p for electrons and protons, n_0 is the density, k is Boltzmann's constant, and T_α is the temperature of each species. To apply the results of Sec. V, we first calculate the function $\kappa^2(v)\equiv A_v^{(2)}$ defined in Eq. (36). The results are depicted in Fig. 9.

For the case $T_e=T_p$ (solid curve in Fig. 9) we find that $\kappa^2(v)>0$ for $v>v_c^{(3)}\simeq 1.3v_e$, where v_e is the electron thermal speed. Thus, from the results above, there exist undamped periodic wave solutions of arbitrarily small amplitude over this velocity region only, i.e., there is a cutoff velocity v_c below which these waves do not exist (a feature already familiar from the linear theory). For $v\gtrsim v_c^{(3)}$, which implies a small wave number, these waves are undamped nonlinear electron-acoustic waves, which satisfy the approximate dispersion relation $\omega\sim 1.3v_e k$ and correspond to the strongly damped electron-acoustic waves of the linear theory. For large v, on the other hand, the relationship between ω and κ becomes the Bohm-Gross relation $\omega^2\sim\omega_p^2+3v_e^2k^2$ for Langmuir waves. This correspondence between undamped periodic waves and Langmuir waves of the linear theory was noted previously [12,13].

When $T_p<\beta T_e\simeq 0.28T_e$, that is, for cold ions, $\kappa^2(v)$ is also positive over the region $v_c^{(1)}<v<v_c^{(2)}$, as indicated schematically by the dashed curve in Fig. 9, where $v_c^{(1)}\simeq 1.3v_p$ and $v_c^{(2)}\simeq C_s=\sqrt{kT_e/m_p}$, v_p denoting the proton thermal speed and C_s the ion-sound speed. For $v\lesssim C_s$ the corresponding waves are exact nonlinear but *undamped* analogs of the weakly damped fast branch of ion-acoustic waves of the linear kinetic theory, the relationship between ω and k being the familiar $\omega\sim C_s k$. For $v\gtrsim v_c^{(1)}$ there exists another acoustic branch of undamped nonlinear waves that satisfy the approximate dispersion relation $\omega\sim 1.3v_p k$ and correspond to the strongly damped slow branch of ion-acoustic waves of the linear theory. Both the fast and slow branches of ion-acoustic waves also can be derived within the context of the nonlinear fluid theory of plasmas [19], where they are of course undamped since, as is well known, the effects of the resonant wave-particle interaction are not manifest in the fluidlike treatment. It is common practice, however, to add artificial damping to the solutions thus obtained so as to give behavior consistent with the expectations of the linear kinetic theory of plasmas; however, the exact nonlinear but undamped periodic waves discussed here indicate that such modifications are not always warranted.

Now let us turn to solitary waves. According to our general results, a generic branch of solitary waves bifurcates from every velocity v_c for which $\kappa^2(v_c)=0$ and $A_{v_c}^{(3)}\neq 0$. Again referring to Fig. 9, which shows κ^2 vs v for a two-species plasma of thermally distributed electrons and protons, we see that when $T_p=T_e$ there is only

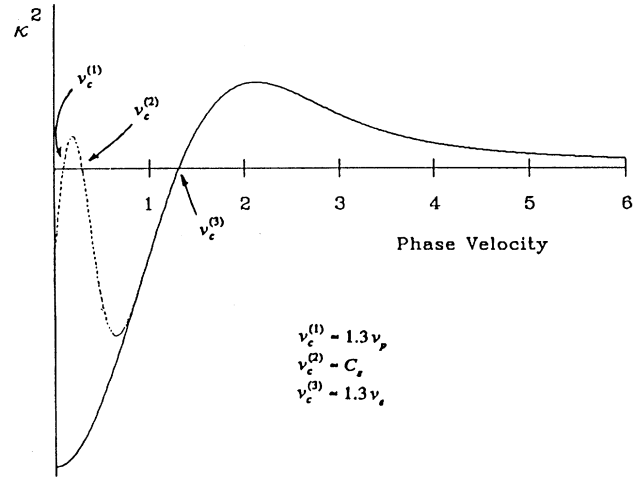

FIG. 9. The solid curve shows the dependence of the parameter κ^2 on the phase velocity v for a thermal e^--p^+ plasma with $T_e=T_p$. Small amplitude nonlinear periodic waves exist for velocities $v>v_c^{(3)}\simeq 1.3v_e$, where $\kappa^2(v)>0$, and small amplitude solitary waves exist for $v\lesssim v_c^{(3)}$, where $\kappa^2(v)\lesssim 0$. If $T_p<\beta T_e\simeq 0.28T_e$, then the shape of $\kappa^2(v)$ changes at lower velocities, as indicated by the dashed curve, in which case small amplitude nonlinear periodic waves also exist for $v_c^{(1)}<v<v_c^{(2)}$, where $\kappa^2(v)>0$ also, and small amplitude solitary waves exist for $v\lesssim v_c^{(1)}$ and $v\gtrsim v_c^{(2)}$, where $\kappa^2(v)\lesssim 0$.

one such velocity, $v_c^{(3)} \simeq 1.3 v_e$. Since $\kappa^2 < 0$ for $v < v_c^{(3)}$ in this case, a branch of solitary waves of the form of Eq. (59) exists for v in the neighborhood of, but less than, $v_e^{(3)}$. When $T_p < \beta T_e \simeq 0.28 T_e$ there are two additional roots of κ^2, or critical velocities, one at $v_c^{(1)} \simeq 1.3 v_p$ and the other at the ion-sound speed $v_c^{(2)} = C_s$. The solitary waves bifurcating from C_s exist for $v \gtrsim C_s$ and are exact nonlinear but undamped kinetic theory analogs of the well-known ion-acoustic solitary waves. Those solutions are usually obtained in the fluid approximation [18,19] and, in fact, the mechanical potential arising in that analysis (sometimes referred to as the Sagdeev potential) is obtained in the present formalism as the lowest-order term in an asymptotic expansion of $A_v(\varphi)$ in the temperature ratio $\tau = T_p / T_e$. There is of course a third branch of solitary waves associated with the root of $\kappa^2(v)$ located at $1.3 v_p$.

Since the fluid theory does not include wave-particle resonance effects, it is common practice, as in the case of periodic waves, to add artificial damping to the solitary waves obtained within the fluid approximation so as to obtain the properties expected from Landau's linearized kinetic theory analysis of periodic waves. While this leads to accurate results in many instances, such as when the initial perturbation does not appreciably alter the background distribution, the foregoing developments show that it is not always appropriate. The existence of small but constant amplitude ion-acoustic solitary waves has been suggested previously by Meiss and Morrison [20].

VII. DEPENDENCE OF $\kappa^2(v)$ ON THE UNDERLYING EQUILIBRIUM

We have seen that small amplitude solitary waves or double layers can exist in a collisionless plasma if their velocity is near a critical velocity given by a root of the condition $\kappa^2(v) = 0$. A natural question then is: Do such roots exist for physically reasonable Vlasov equilibria? That the answer is "yes" was indicated in Sec. VI, where we displayed $\kappa^2(v)$ for the specific and important e^--p^+ thermal plasma. It is possible, however, to formulate a more general statement. Recall that the parameter $\kappa^2(v)$ is calculated from the Vlasov equilibrium distribution functions $F_\alpha(u)$ as

$$\kappa^2(v) = 4\pi \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int_{-\infty}^{\infty} dw \frac{1}{w} \frac{dF_\alpha^{v,e}}{dw} . \tag{70}$$

Integrating this expression over all values of the phase velocity v gives an interesting result

$$\begin{aligned}\int_{-\infty}^{\infty} dv\, \kappa^2(v) &= 4\pi \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int dv \int dw \frac{1}{w} \frac{d}{dw} F_\alpha^{v,e}(w) \\ &= 4\pi \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int dv \frac{d}{dv} \int dw \frac{1}{w} F_\alpha^{v,e}(w) \\ &= 0 .\end{aligned} \tag{71}$$

Now since $\kappa^2(v)$ is a continuous function of v (and is not identically zero), we may immediately conclude from Eq. (71) that $\kappa^2(v)$ takes on both positive and negative values and therefore passes through zero at least once. From this result, in conjunction with the condition derived in Sec. V A, it follows that small amplitude traveling periodic waves, which neither damp nor grow, exist near a very broad class of Vlasov equilibria. Furthermore, as shown in Secs. V B and V C, the existence of a velocity v_c at which $\kappa^2(v)$ vanishes implies, in general, the existence of small amplitude traveling solitary waves or traveling double layers with velocity v approximately equal to v_c. Thus the above result implies that most physically relevant plasma equilibria support spatially periodic traveling waves as well as traveling solitary or double-layer waves of small but constant amplitude.

VIII. NONLINEAR LANDAU DAMPING

In the previous sections we have described a broad spectrum of constant amplitude near-equilibrium BGK plasma waves. So far, however, we have not presented any evidence that such waves are truly relevant in physical plasmas, nor have we discussed possible dynamical processes through which such waves might be formed. One such process is the collisionless damping of a finite amplitude plasma wave, so-called nonlinear Landau damping, which was first analyzed by O'Neil [21]. Consider the evolution of an electron plasma (neutralized by a homogeneous background of positive charge) from an initial state given by

$$f(x,u,0) = [1 - (kE_i / 4\pi e n_0) \cos(kx)] F(u) , \tag{72}$$

$$E(x,0) = E_i \sin(kx) , \tag{73}$$

where E_i is the initial amplitude of the electric field and $F(u)$ is a linearly stable electron distribution satisfying $n_0 = \int du\, F(u)$. O'Neil showed that the evolution of the main wave (wave number k) depends crucially upon the ratio γ_L / ω_B, where γ_L is the damping coefficient calculated from linear plasma kinetic theory and $\omega_B = (ekE_i / m)^{1/2}$ is the "bounce" frequency for trapped electrons, i.e., the oscillation frequency for electrons located near the bottom of the potential wells of the wave. For $\gamma_L / \omega_B \gg 1$, the wave damps away completely as predicted by the linear theory; conversely, if $\gamma_L / \omega_B \ll 1$, the linear theory breaks down after a time of the order $\tau = 2\pi / \omega_B$ due to particle trapping. In this latter case, the wave amplitude exhibits damped oscillations and eventually saturates at a finite final amplitude. Importantly, in the limit $\gamma_L \to 0$ the condition $\gamma_L / \omega_B \ll 1$ can be satisfied even for an arbitrarily small initial field amplitude E_i; O'Neil's analysis therefore implies that the linear theory can fail even for plasmas that are arbitrarily close to equilibrium.

In the case $\gamma_L / \omega_B \ll 1$, the plasma evolves toward a state containing finite amplitude waves. If the equilibrium $F(u)$ is symmetric in velocity, then the time-asymptotic state cannot contain only a single wave, since this would violate the space-reflection symmetry of the initial conditions Eqs. (72) and (73), which is preserved by the dynamics of the Vlasov-Poisson system. In fact, recent numerical simulations [22,23] strongly suggest that the plasma evolves toward a state containing a superposition of two counterpropagating BGK waves of the type developed here. For instance, in simulations of the Vlasov-Poisson system with periodic boundary conditions, Demieo and Zweifel [22] chose k in order to stimulate the longest wavelength mode for the system, which, when $F(u)$ is Maxwellian, is also the most weakly damped. Using an initial amplitude sufficiently large for particle trapping to predominate before wave damping is complete, their results show that, while a small quantity of energy leaks into other modes, the main electric field mode has much larger amplitude than any other throughout the entire process. After initially damping in agreement with the linear theory and then exhibiting the damped amplitude oscillations predicted by O'Neil [21], the field finally settles into a standing wave pattern, which indicates a superposition of two counterpropagating waves of equal amplitude and speed. The simulation also shows that the electron distribution function forms two phase space vortices centered at velocities $\pm v_p$, where v_p is the wave phase velocity. Such vortices appear to correspond to particles that have become trapped in the electric potentials of the two counterpropagating waves. In fact, all the numerical evidence suggests strongly that the asymptotic state is well described by two superimposed small amplitude undamped BGK waves that propagate in opposite directions with equal speeds and amplitudes. Moreover, we recently have also reported analytical solutions representing such superpositions of small amplitude spatially periodic BGK waves [17,24,25]. The simulation described in Ref. [22] was followed for a long time without any significant further change apparent in the state of the plasma suggesting that the final state to which the plasmas evolved corresponded to a superposition of the spatially periodic wave solutions discussed here in Sec. V A.

IX. SUMMARY AND CONCLUSIONS

In this paper we have studied the Vlasov-Poisson-Ampère system of equations that provides the most appropriate theoretical model for one-dimensional electrostatic processes in collisionless plasmas under nonrelativistic conditions. Any set of distribution functions $F_\alpha(u)$ that yields vanishing charge and current densities characterizes a spatially uniform field-free equilibrium solution of this model—a Vlasov equilibrium—and corresponds, insofar as binary collisions are neglected, to an equilibrium state (stable or unstable) of the physical plasma.

As reviewed in Sec. III, much of what we know about small amplitude processes near Vlasov equilibria follows from Landau's classic analysis of the Vlasov-Poisson system linearized about an equilibrium $F_\alpha(u)$. It is generally accepted that the description offered by the linear theory is incorrect for a plasma that is sufficiently far from equilibrium. In many cases, however, the linear theory does not give an adequate description even for plasmas that are arbitrarily close to equilibrium. Clearly, the validity of the linear approximation rests upon the condition

$$|\partial h_\alpha/\partial u| \ll |dF_\alpha/du| \; , \tag{74}$$

where $h_\alpha = f_\alpha - F_\alpha$. But, as we have seen, this condition is not an automatic consequence of the smallness of either h_α or φ. In fact, there exist nonlinear traveling wave solutions of arbitrarily small amplitude that do not exhibit damping or growth, even when the linear theory suggests that they should. These nonlinear wave solutions were first discussed by Bohm and Gross, although we have called them BGK waves to agree with the prevailing literature where they are associated with the later work of Bernstein, Greene, and Kruskal. The existence of undamped waves arbitrarily close to Vlasov equilibria is also implicit in O'Neil's seminal work [21] on nonlinear Landau damping. The distinctive feature of BGK waves, as opposed to those described by the linear theory, is that some of the plasma particles are trapped within the potential wells formed by the electric potential $\varphi(\xi)$.

A close analysis [13] shows, in fact, that the linear theory has no implications concerning the properties of these waves. Due to particle trapping, the distribution functions $f_\alpha = F_\alpha + h_\alpha$ must necessarily satisfy the condition $(\partial f_\alpha/\partial u)|_{u=v} = 0$, or equivalently $(\partial h_\alpha/\partial u)|_{u=v} = -(dF_\alpha/du)|_{u=v}$, which violates the condition of Eq. (74) for the applicability of the linear approximation even as the wave amplitude approaches zero. The theory of undamped waves has previously been treated with increasing detail, rigor, and generality by Bohm and Gross; Bernstein, Greene, and Kruskal; and finally by Holloway and Dorning. In Sec. V, after introducing the mechanical potential formalism, which greatly simplifies much of the required analysis, we then used this formalism to investigate methodically the types and properties of small amplitude BGK waves, both spatially periodic and aperiodic, that can exist nearby a plasma equilibrium characterized by a given set of distribution functions. This analysis showed that, in addition to spatially periodic waves that satisfy the original dispersion relation of Vlasov in the small amplitude limit, there also exists a broad class of solitary BGK waves of arbitrarily small but constant amplitude. Specifically, we found that any collisionless plasma is characterized by a discrete set of critical velocities $v_c^{(i)}, i = 1, 2, \ldots,$ which are the velocities at which BGK solitary waves of vanishingly small amplitude can propagate through the plasma. These critical velocities are determined as the roots of the fundamentally important function $\kappa^2(v)$, which is determined uniquely by the distribution functions $F_\alpha(u)$ of the underlying equilibrium. In addition, we also showed that charge-symmetric plasmas such as the electron-positron plasma can support slightly more exotic types of solutions—traveling double layers or kinks—which represent traveling transition regions, or buffer zones, between different homogeneous portions of the plasma with unequal electric potential.

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APPENDIX A: ODD FUNCTIONS $g_\alpha^{v,o}$

As mentioned in Sec. IV B, the definition of $g_\alpha^{v,o}$ given by Eq. (32) is in general not suitable since the trapping of particles in the electrostatic potential of the wave requires the distribution functions $f_\alpha(\xi,w)$ to be even functions of the velocity w in the neighborhood of $w=0$ ($u=v$). Therefore, the definition must be modified so as to satisfy

$$g_\alpha^{v,o}(\mathcal{E}_\alpha)=0, \quad \mathcal{E}_\alpha \le Q_\alpha \equiv |q_\alpha \varphi|_{\max} , \tag{A1}$$

which means that the odd parts of the distribution functions depend upon the amplitude parameter φ_0 and vanish over the trapping regions. In this paper we only consider cases in which the variable components $h_\alpha^{v,e}$ of the even parts of the distribution functions are chosen in the particularly simple from $h_\alpha^{v,e}=\mu_\alpha G_\alpha^{v,e}$, where the μ_α are a set of real parameters. For any branch of solutions $(\varphi(\xi;\varphi_0),\mu_\alpha(\varphi_0))$ of Eqs. (15) and (16) obtained by the methods of Sec. V, a prescription for the modified functions $g_\alpha^{v,o}$ has been given in Ref. [13] as

$$\begin{aligned} g_\alpha^{v,o}(\mathcal{E}_\alpha,\varphi_0)=&[1+\mu_\alpha(\varphi_0)][1-R(\mathcal{E}_\alpha/Q_\alpha)]\\ &\times\{[1-\beta_\alpha(\varphi_0)]G_\alpha^{v,o}(\mathcal{E}_\alpha)\\ &+\beta_\alpha(\varphi_0)G_\alpha^{v,e}(\mathcal{E}_\alpha)\} ,\end{aligned} \tag{A2}$$

where

$$R(\eta)=\begin{cases} 1, & \eta\le 1\\ \dfrac{1}{2}\left[1-\tanh\left[\dfrac{3-2\eta}{2(\eta-1)(\eta-2)}\right]\right], & 1\le\eta\le 2\\ 0, & \eta\ge 2\end{cases} \tag{A3}$$

and the quantities $\beta_\alpha(\varphi_0)$, the importance of which is discussed below, are constants for any particular value of φ_0. Since the function $R(\eta)$ is infinitely differentiable, the factor $1-R(\mathcal{E}_\alpha/Q_\alpha)$ cuts off $g_\alpha^{v,o}$ in a smooth way so that it satisfies Eq. (A1) above. When $g_\alpha^{v,o}$ is substituted into Ampère's equation, Eq. (18), we then have

$$0=8\pi\sum_\alpha q_\alpha\int_0^\infty dw\, w\, g_\alpha^{v,o}(\tfrac{1}{2}m_\alpha w^2+q_\alpha\varphi(\xi)) , \tag{A4}$$

which is a zero current constraint in the wave frame. It is straightforward to verify that if the functions $\beta_\alpha(\varphi_0)$ in Eq. (A2) are defined as

$$\beta_\alpha(\varphi_0)=\frac{\int_0^\infty d\mathcal{E}_\alpha R(\mathcal{E}_\alpha/Q_\alpha)G_\alpha^{v,o}(\mathcal{E}_\alpha)}{\int_0^\infty d\mathcal{E}_\alpha[1-R(\mathcal{E}_\alpha/Q_\alpha)][G_\alpha^{v,e}(\mathcal{E}_\alpha)-G_\alpha^{v,o}(\mathcal{E}_\alpha)]}, \tag{A5}$$

then the above constraint is satisfied, while, in addition, the overall distribution functions $f_\alpha=f_\alpha^e+f_\alpha^o$ are nonnegative. Furthermore, the odd parts of the distribution functions $f_\alpha^o(\xi,w)=g_\alpha^{v,o}(\mathcal{E}_\alpha)$ uniformly approach the odd parts of the velocity-shifted equilibrium $F_\alpha^{v,o}(w)$ as φ_0 or, equivalently, the wave amplitude goes to zero. The prescription for $g_\alpha^{v,o}$ given above is not unique; on the contrary, there is an infinite number of ways to define the $g_\alpha^{v,o}$ so as to satisfy both Eqs. (A1) and (A4). Importantly, however, this result establishes the fact that corresponding to any near-equilibrium solution of Eqs. (15) and (16) is at least one and usually many physically reasonable solutions of Eqs. (17) and (18).

APPENDIX B: WAVES WITH CONSTANT PARTICLE DENSITY

In this appendix we shall construct a branch of wave solutions for which the average particle density is the same as that of the underlying equilibrium [17]. We consider waves at phase velocity v for which $\kappa^2(v)>0$. Once again, the nonlinear differential equation to be solved for the electric potential is Eq. (37) where $A_v(\varphi)=A_{v,h}(\varphi)$ depends parametrically upon the variable functions $h_\alpha^{v,e}$ as indicated in Eqs. (48) and (49). The final equation to be solved then is Eq. (50) in which the mechanical potential is expanded for small $\varphi(\xi)$. For this example we again use the particularly simple form for the functions $h_\alpha^{v,e}$, namely, $h_\alpha^{v,e}=\mu_\alpha G_\alpha^{v,e}$, where each μ_α is an independent real variable. We shall find that by varying these μ_α appropriately with the amplitude of the wave solution we can construct a family of undamped waves of velocity v for which the average particle density of each species remains constant irrespective of the (small) wave amplitude. We demonstrate the calculation only through second order in the wave amplitude, although it is straightforward to continue to higher orders.

When $\mu_\alpha=0$ for all species we have by construction $\kappa^2(v)>0$ and thus $A_v(\varphi)$ has a local minimum at $\varphi=0$ that guarantees the existence of small amplitude periodic solutions of the approximate form $\varphi_0\cos k\xi$, corresponding to boundary conditions $[\varphi(0)=\varphi_0,\ (d\varphi/d\xi)(0)=0]$, where φ_0 is the small wave amplitude. The wave number k depends in general upon φ_0, i.e., $k=k(\varphi_0)$, as do the spatially averaged particle densities

$$\langle n_\alpha\rangle=\frac{k}{2\pi}\int_0^{2\pi/k}d\xi\int dw\, G_\alpha^{v,e}\left[\frac{1}{2}m_\alpha w^2+\frac{q_\alpha}{m_\alpha}\varphi(\xi)\right] . \tag{B1}$$

As $\varphi_0\to 0$, $k(\varphi_0)$ approaches $\kappa(v)$ while the densities $\langle n_\alpha\rangle$ approach their equilibrium values $n_{\alpha,0}$.

Now suppose that the parameters μ_α are no longer zero, in which case the mechanical potential depends upon the μ_α as well as φ and takes the form of Eq. (48). As a result, the wave number k and the average particle densities $\langle n_\alpha\rangle$, which correspond to a periodic solution of amplitude φ_0, now also depend upon the μ_α, i.e., $k=k(\varphi_0,\mu_\alpha)$, $\langle n_\alpha\rangle=\langle n_\alpha\rangle(\varphi_0,\mu_\alpha)$. By varying the μ_α appropriately along with φ_0, we can force the densities $\langle n_\alpha\rangle$ to retain their equilibrium values $n_{\alpha,0}$ independently of φ_0. To see this, at least for small φ_0, we shall write the connection between the μ_α and φ_0 as $\mu_\alpha=\mu_\alpha^{(1)}\varphi_0+\mu_\alpha^{(2)}\varphi_0^2+O(\varphi_0^3)$ and determine the constants $\mu_\alpha^{(1)}$ and $\mu_\alpha^{(2)}$ by calculating $\langle n_\alpha\rangle$ and then requiring $\langle n_\alpha\rangle=n_{\alpha,0}$.

We first must solve Eq. (50) for the electric potential $\varphi(\xi)$ of the small amplitude wave solutions. When the quantities $\delta A^{(i)}$ appearing in Eq. (50) do not vanish, the solutions $\varphi(\xi)$ will be centered about the nonzero value $\varphi_{\min}$ given in Eq. (51). It is convenient therefore to rewrite Eq. (50) in terms of a new variable $\chi(\xi)$, which is equal to $\varphi(\xi)$ less the quantity $\varphi_{\min}$, $\chi(\xi)=\varphi(\xi)-\varphi_{\min}$. It is then easy to show that the new variable $\chi(\xi)$ satisfies the equation

$$\frac{d^2\chi}{d\xi^2}=-[\kappa^2(v)+O(\delta A)]\chi-\tfrac{1}{2}[A_v^{(3)}+O(\delta A)]\chi^2 -\frac{1}{3!}[A_v^{(4)}+O(\delta A)]\chi^3+O(\chi^4)\ , \tag{B2}$$

where $O(\delta A)$ denotes terms of order $\delta A^{(i)}$ and $O(\delta A)=O(\varphi_0)$ by virtue of the assumed form $\mu_\alpha=\mu_\alpha^{(1)}\varphi_0+\mu_\alpha^{(2)}\varphi_0^2+O(\varphi_0^3)$ and the definition of Eq. (49).

The explicit solutions to Eq. (B2) can be obtained via the method of Poincaré and Lindstedt, a standard technique in perturbation theory [26]. Denoting the (small) amplitude parameter as φ_0, then the solution corresponding to the zeroth order boundary conditions $[\varphi(0)=\varphi_0, \varphi_\xi(0)=0]$ may be developed in series form, resulting in the expression

$$\chi(\xi)=\varphi_0\cos(k\xi)-\frac{A_v^{(3)}}{4\kappa^2(v)}\varphi_0^2[1-\tfrac{1}{3}\cos(2k\xi)] +\frac{(A_v^{(3)})^2}{192\kappa^4(v)}\varphi_0^3\cos(3k\xi)+O(\varphi_0^4)\ , \tag{B3}$$

where the wave number k also depends upon the amplitude parameter as

$$k=\kappa(v)+\frac{1}{2\kappa(v)}\left[\frac{3}{24}A_v^{(4)}-\frac{5}{24}\frac{(A_v^{(3)})^2}{\kappa^2(v)}\right]\varphi_0^2+O(\varphi_0^3)\ . \tag{B4}$$

Thus $\varphi(\xi)=\chi(\xi)+\varphi_{\min}$ gives a branch of small amplitude periodic wave solutions (traveling waves of velocity v in the original frame of reference) parametrized by the amplitude φ_0.

We now use the distribution functions corresponding to these waves to calculate the average particle densities

$$\langle n_\alpha\rangle=\frac{k}{2\pi}\int_0^{2\pi/k}d\xi\int dw\, f_\alpha^e(\xi,w)\ , \tag{B5}$$

where, as indicated, $\langle n_\alpha\rangle$ depends only on the even parts of the distribution functions $f_\alpha^e(\xi,w)$. Since these are given in the BGK representation as $f_\alpha^e(\xi,w)=(1+\mu_\alpha)G_\alpha^{v,e}(\mathcal{E}_\alpha)$, we have

$$\langle n_\alpha\rangle=(1+\mu_\alpha)\frac{k}{2\pi}\int_0^{2\pi/k}d\xi\int dw\, G_\alpha^{v,e}(\tfrac{1}{2}m_\alpha w^2 +q_\alpha\varphi(\xi))\ . \tag{B6}$$

Upon expanding the integrand in this expression in powers of $\varphi(\xi)$ and performing the indicated integrations, we obtain

$$\langle n_\alpha\rangle=[1+\mu_\alpha(\varphi_0)][n_{\alpha,0}+q_\alpha D_{\alpha,1}\langle\varphi(\xi)\rangle +\tfrac{1}{2}q_\alpha^2D_{\alpha,2}\langle\varphi^2(\xi)\rangle+O(\varphi_0^3)]\ , \tag{B7}$$

where $\langle\ \rangle$ denotes the spatial average, $n_{\alpha,0}=\int dw\, G_\alpha^{v,e}(m_\alpha w^2/2)$ are the equilibrium particle densities, and we have introduced the quantities

$$D_{\alpha,1}(v)=\int dw\frac{dG_\alpha^{v,e}}{d\mathcal{E}_\alpha}=\frac{1}{m_\alpha}\int dw\frac{1}{w}\frac{d}{dw}F_\alpha^{v,e}(w) \tag{B8}$$

and

$$D_{\alpha,2}(v)=\int dw\frac{d^2G_\alpha^{v,e}}{d\mathcal{E}_\alpha^2} =\frac{1}{m_\alpha^2}\int dw\frac{1}{w}\frac{d}{dw}\left[\frac{1}{w}\frac{d}{dw}F_\alpha^{v,e}(w)\right]\ . \tag{B9}$$

The spatial averages $\langle\varphi(\xi)\rangle$ and $\langle\varphi^2(\xi)\rangle$ may be calculated from Eqs. (B3), (51), and (41). Inserting those averages and the above expressions along with $\mu_\alpha(\varphi_0)=\mu_\alpha^{(1)}\varphi_0+\mu_\alpha^{(2)}\varphi_0^2+O(\varphi_0^3)$ into Eq. (B7) gives the final result for $\langle n_\alpha\rangle$ explicitly as a function of φ_0, from which we can determine $\mu_\alpha^{(1)}$ and $\mu_\alpha^{(2)}$. If $\langle n_\alpha\rangle$ is to be equal to $n_{\alpha,0}$ independently of φ_0, then the coefficients of the terms in $\langle n_\alpha\rangle$ proportional to φ_0 and φ_0^2 must vanish. By setting each of these coefficients equal to zero, we therefore obtain two conditions involving $\mu_\alpha^{(1)}$ and $\mu_\alpha^{(2)}$. The first of these is

$$\mu_\alpha^{(1)}n_{\alpha,0}-\frac{4\pi}{\kappa^2}q_\alpha D_{\alpha,1}\sum_\alpha q_\alpha\mu_\alpha^{(1)}n_{\alpha,0}=0\ , \tag{B10}$$

which, being homogeneous in the parameters $\mu_\alpha^{(1)}$, can be satisfied trivially by choosing $\mu_\alpha^{(1)}=0$ for all α. This reflects the fact that the average density $n_{\alpha,0}$ has only a second-order dependence on the wave amplitude φ_0.

At second order in φ_0 we find, when $\mu_\alpha^{(1)}=0$,

$$\mu_\alpha^{(2)}n_{\alpha,0}+\tfrac{1}{4}q_\alpha^2D_{\alpha,2} -q_\alpha D_{\alpha,1}\left[\frac{4\pi}{\kappa^2}\sum_\alpha q_\alpha\mu_\alpha^{(2)}n_{\alpha,0}+\frac{A_v^{(3)}}{4\kappa^2}\right]=0\ . \tag{B11}$$

By inspection this has the solution $\mu_\alpha^{(2)}=-q_\alpha^2D_{\alpha,2}/4n_{\alpha,0}$, since in this case the term in large parentheses vanishes as a result of the equality $\sum_\alpha q_\alpha\mu_\alpha^{(2)}n_{\alpha,0}=-\frac{1}{4}\sum_\alpha q_\alpha^3D_{\alpha,2}=-A_v^{(3)}/16\pi$, which follows from Eq. (B9) and the definition of $A_v^{(3)}$. Thus, if we choose the variable functions $h_\alpha^{v,e}$ as $h_\alpha^{v,e}=\mu_\alpha(\varphi_0)G_\alpha^{v,e}$, where $\mu_\alpha(\varphi_0)=-(q_\alpha^2D_{\alpha,2}/4n_{\alpha,0})\varphi_0^2+O(\varphi_0^3)$, then the small amplitude spatially periodic wave solutions with electric potential $\varphi(\xi)=\chi(\xi)+\varphi_{\min}$, where $\chi(\xi)$ is given in Eq. (B3) and $\varphi_{\min}$ by Eq. (51), describe plasma states that, to second order in φ_0, have the same average number of particles of each species as the equilibrium that they are near. In the original frame of reference these are traveling waves that could be excited by a plasma perturbation, such as the application of an electrostatic field, which does not alter the number of particles in the plasma through second order in φ_0. It is straightforward to continue the above procedure to higher orders.

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